Rice's theorem for generic limit sets of cellular automata

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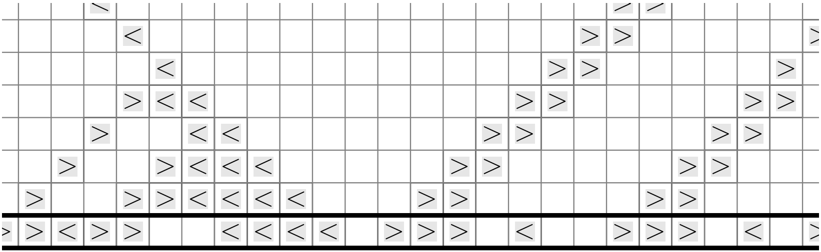
Context



- Only 1D-CA here.
- The *limit set* is the set of all asymptotic behaviours.
- The *generic limit set* (Milnor 1985) is a topological approach of an asymptotic set of typical configurations .

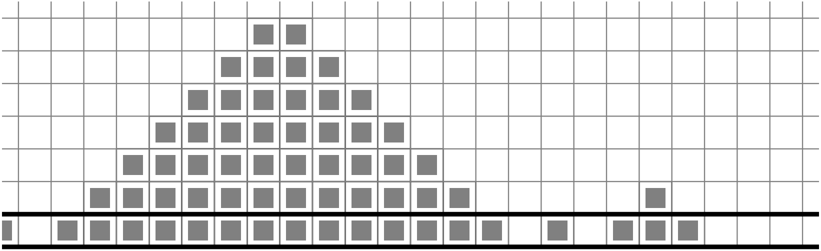
Definitions and first results

Cellular Automata (CA)

A one-dimensional CA $\mathfrak{F}$ is given by:

- a finite alphabet Σ
- a radius $r \in \mathbb{N}$
- a local rule $\delta : \Sigma^{2r+1} \to \Sigma$

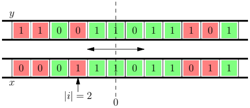
Starting from an initial configuration $c \in \Sigma^{\mathbb{Z}}$, the global rule associated to δ can be applied successively to the images $\mathfrak{F}^n(c)$ for $n \in \mathbb{N}$. We represent the orbit as the pile of configurations (time going up) called the *space-time diagram*.



Topology

We define a *distance* on the set $\Sigma^{\mathbb{Z}}$ of all configurations by :

$$\forall x, y \in \Sigma^{\mathbb{Z}}, d(x, y) = 2^{-min\{|i| : x_i \neq y_i\}}$$



It is the *Cantor* topology and the balls are called cylinders. For a word $u \in \Sigma^*$ and a position $i \in \mathbb{Z}$, define:

$$[u]_i = \{c \in \Sigma^{\mathbb{Z}} : c_{[i..i+|u|-1]} = u\}$$

We will often write $[u]$ for $[u]_0$.
$\Sigma^{\mathbb{Z}}$ is compact and the cylinders form a basis of clopen sets.

Shift and subshifts

Define the shift $\sigma : \Sigma^{\mathbb{Z}} \to \Sigma^{\mathbb{Z}}$ with $\sigma(c)_i = c_{i+1}$ for any configuration c and position i.

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We are especially interested in the set of closed and σ-invariant subsets of $\Sigma^{\mathbb{Z}}$, they are called *subshifts* and can be equivalently defined by a set of forbidden patterns $\mathcal{F}$:

$$X_{\mathcal{F}} = \{c \in \Sigma^{\mathbb{Z}} : \forall i \in \mathbb{Z}, \forall u \in \mathcal{F}, c_{i..i+|u|-1} \neq u\}$$

The limit set

The *limit set* is the set of configurations that have antecedents arbitrarily far back in time.

$$\Omega(\mathfrak{F}) = \bigcap_{n \in \mathbb{N}} \mathfrak{F}^n(\Sigma^{\mathbb{Z}})$$

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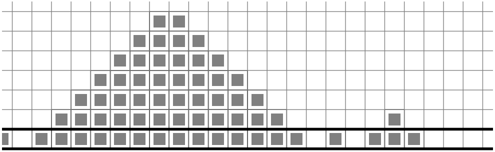

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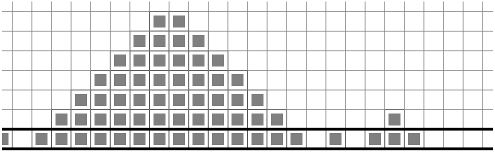

$\Omega(min)$ contains every configuration where ▣□^{k}▣ does not appear:

- ...□□□□□□□□□□... , ...▣▣▣▣▣▣▣▣▣▣...
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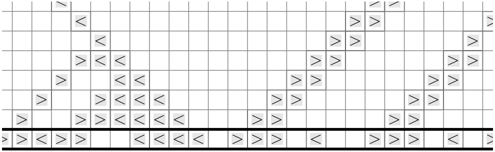

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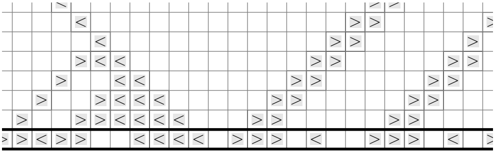

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The realm of attraction

For any subset $X \subseteq \Sigma^{\mathbb{Z}}$, define $\omega(\mathfrak{F})(X)$ as the set of limit points of orbits of configurations in X:

$$c \in \omega(\mathfrak{F})(X) \Leftrightarrow \exists c' \in X, \liminf_{t\to\infty} d(\mathfrak{F}^t(c'), c) = 0$$

For $X \subseteq \Sigma^{\mathbb{Z}}$, define the *realm of attraction*

$$\mathcal{D}(X) = \{c \in \Sigma^{\mathbb{Z}} : \omega(\mathfrak{F})(c) \subseteq X\}$$

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Comeager set

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- Its realm is comeager, hence $\tilde{\omega}(\mathfrak{F})$ is nonempty.
- $\tilde{\omega}(\mathfrak{F}) = \Sigma^{\mathbb{Z}} \Leftrightarrow \mathfrak{F}$ is surjective.

The generic-limit set, characterization

Using a property proved in [Djenaoui-Guillon 2018], Törmä (2020) proved a combinatorial characterization of generic limit sets.

Lemma [Törmä 2020]

Let $\mathfrak{F}$ be a CA on $\Sigma^{\mathbb{Z}}$. A word $s \in \Sigma^*$ belongs to $L(\tilde{\omega}(\mathfrak{F}))$ if and only if there exists a word $v \in \Sigma^*$ and $i \in \mathbb{Z}$ such that for all $u, w \in \Sigma^*$, there exist infinitely many $t \in \mathbb{N}$ with $\mathfrak{F}^t([uvw]_{i-|u|}) \cap [s] \neq \emptyset$. Such a word v is said to enable s.

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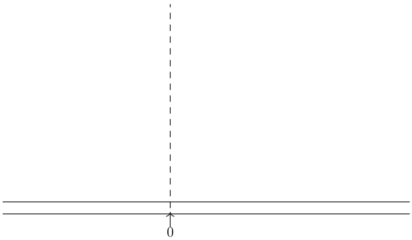

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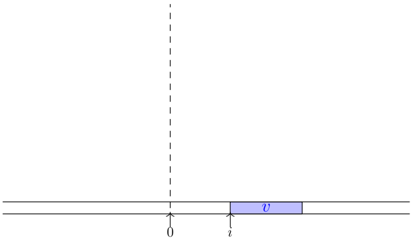

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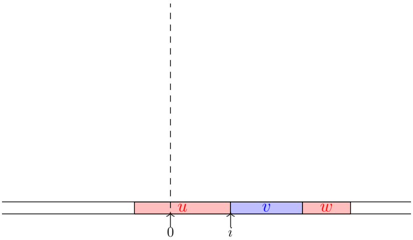

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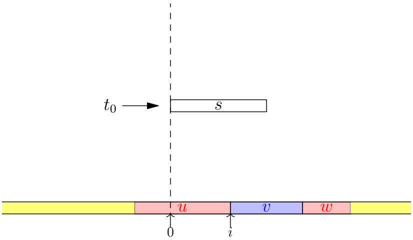

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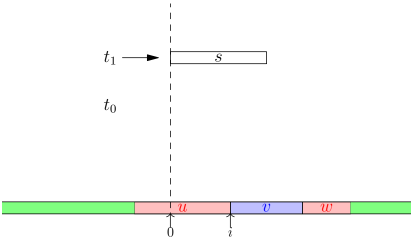

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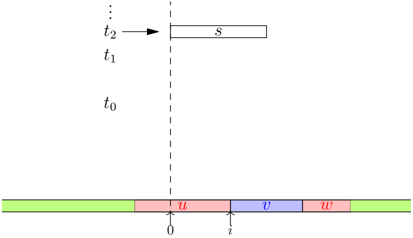
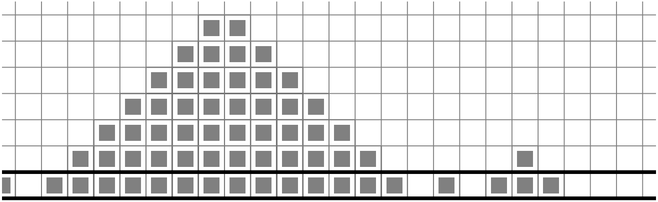


The generic-limit set, examples

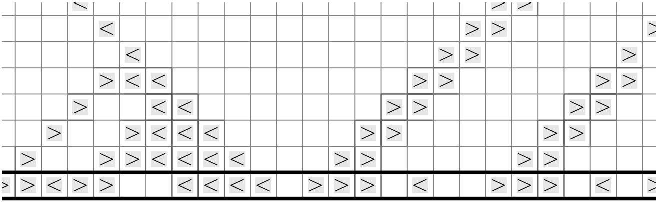


$\tilde{\omega}(min)$ contains only the uniform configuration:

$$\ldots\square\square\square\square\square\square\square\square\square\square\ldots$$

- The empty word enables 0.
- There is no word that enables 1.

The generic-limit set, examples



$\tilde{\omega}(glid)$ is equal to the limit set $\Omega(glid)$ (all the configurations where $\boxed{<}\boxed{\phantom{<}}^{k}\boxed{>}$ does not appear).

Comparisons with a measure-theoretical approach

	μ-limit $\Lambda_\mu(\mathfrak{F})$	generic $\tilde{\omega}(\mathfrak{F})$	limit $\Omega(\mathfrak{F})$
min	$\{\square^{\mathbb{Z}}\}$	$\{\square^{\mathbb{Z}}\}$	$X_{\blacksquare\square^k\blacksquare}$
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It is proved [Djenaoui-Guillon 2018] that

$$\Lambda_\mu(\mathfrak{F}) \subseteq \tilde{\omega}(\mathfrak{F}) \subseteq \Omega(\mathfrak{F})$$

Already known

Finite sets:

- If the limit set is finite, it is a singleton [Culik-Pachl-Yu 1989].
- Generic limit sets can have any finite cardinal.

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The maximal complexity of the language is:

- Π_1-complete for the limit set [Culik-Pachl-Yu 1989].
- Σ_3-complete for the generic limit set [Törmä 2020].

Rice's theorem

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On Turing machines:

[Rice 1953]

Every property of languages of Turing machines is either trivial or undecidable.

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Concerning CA:

Rice's theorem for limit sets of CA [Kari 1994]

Every property of limit sets of CA is either trivial or undecidable.

Rice's theorem for generic limit sets of CA [Delacourt 2021]

Rice's theorem for generic limit sets of CA

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A property is said to be trivial if either it contains all generic limit sets or none.

The proof

Outline

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Take a non trivial property $\mathcal{P}$ of generic limit sets of CA. Assume there exists $q_n \in \mathcal{Q}$ and a CA $\mathfrak{F}_1$ such that:

- $\tilde{\omega}(\mathfrak{F}_1) \notin \mathcal{P}$
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For any Turing machine M, we will produce a CA $\mathfrak{F}_M$ such that:

- if M eventually halts on empty input, the generic limit set of $\mathfrak{F}_M$ is $\{q_n^{\mathbb{Z}}\}$;
- if M never halts on empty input, then the generic limit set of $\mathfrak{F}_M$ is $\tilde{\omega}(\mathfrak{F}_1)$.

Overview of the construction of $\mathfrak{F}_M$

We use two layers:

- the second layer contains a computation of $\mathfrak{F}_1$.
- the first layer contains a structure that will simulate the computation of M in finite areas called segments.
  In every such segment, when the computation is over (due to a time limit or the halting of M):
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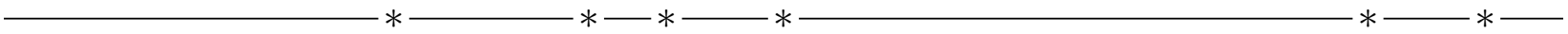
The idea is that we can produce segments of arbitrary large size in almost every space-time diagram. Hence, if M eventually halts, some segments will be large enough to reach this step.

First layer: counters

We use one particular state $\boxed{*}$ that can appear only in the initial configuration.



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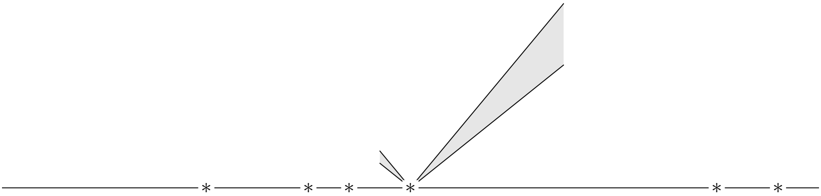

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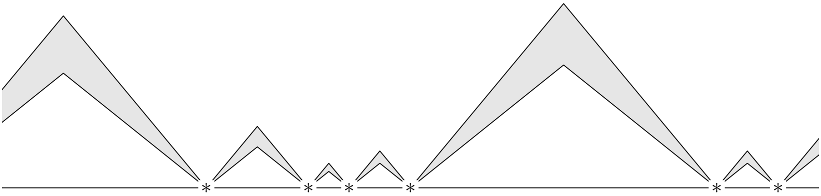

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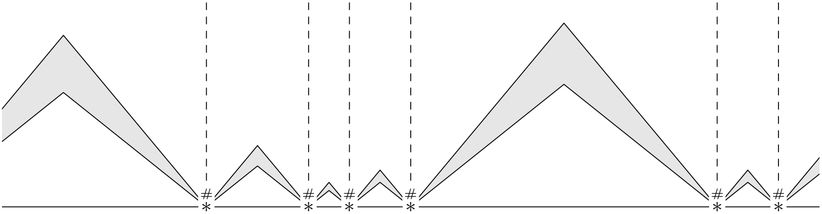
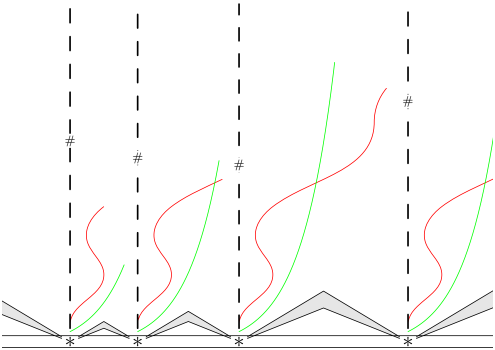


First layer: segments

The segments are the set of cells between two consecutive $\boxed{\#}$. In each of them a simulation of M on empty input is started. In parallel a binary counter starts to increment.



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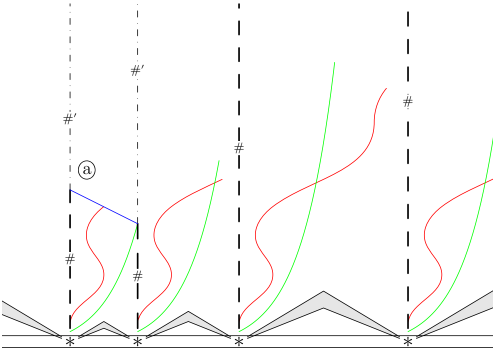

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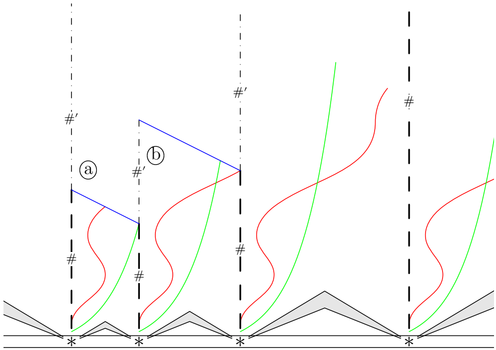

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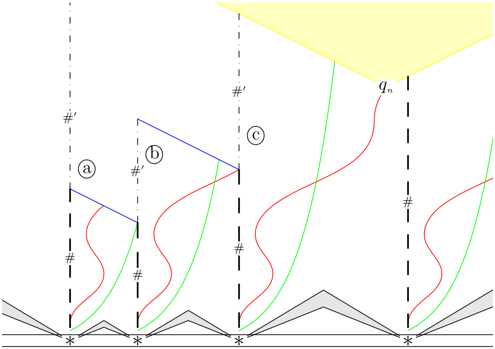
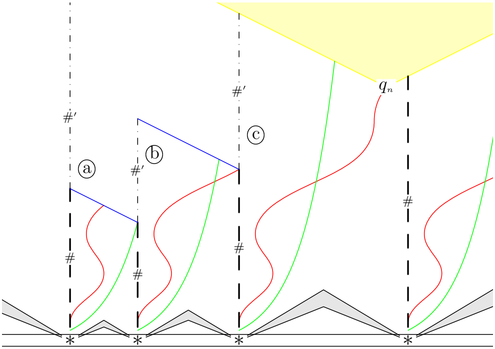
- case ⓐ: the counter reaches the other side of the segment.
- case ⓑ: the MT reaches the other side of the segment.
- case ⓒ: the computation of M ends.



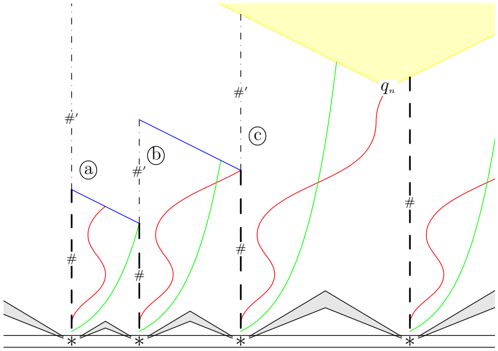
Cases ⓐ and ⓑ

- In both cases the content of the first layer is erased in the segment.



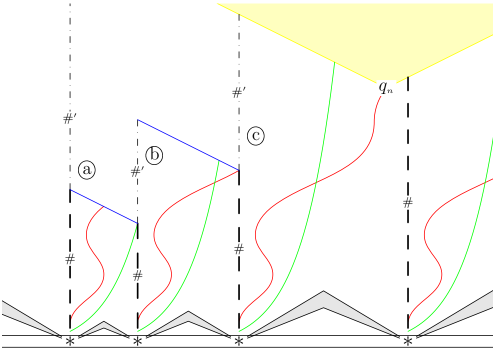
Cases ⓐ and ⓑ

- In both cases the content of the first layer is erased in the segment.
- The [#] states are also erased when both segments are.



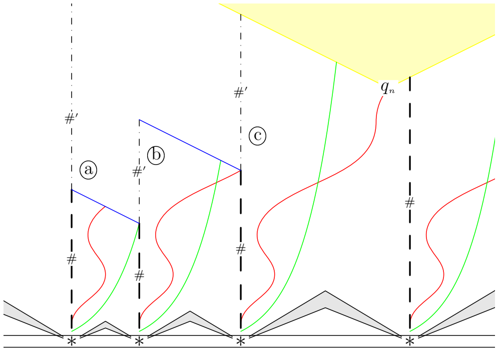
Case ⓒ

- In this case the contents of both layers are erased and state q_n is written.



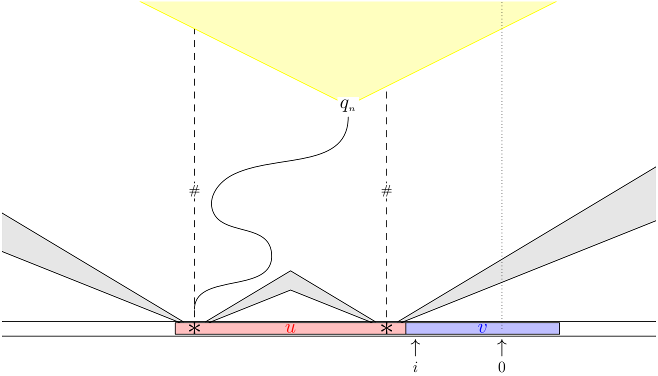
Case ⓒ

- In this case the contents of both layers are erased and state q_n is written.
- q_n is spreading, so it invades the whole configuration.



Behaviour of the first layer

If M halts on empty input, there will exist a large enough segment to reach the end of the computation of M in almost every configuration. In this case, the state q_n is the only one that remains in the generic limit set. (It is impossible to enable any other state than q_n.)



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If M does not halt on empty input, the first layer of every segment will eventually be erased and the second layer will remain, hence $\mathfrak{F}_M$ will tend to act as $\mathfrak{F}_1$.

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Hence any word enabled for $\mathfrak{F}_1$ will be enabled for $\mathfrak{F}_M$.

Initializing the second layer

The proof works as long as the second layer performs indeed a “reasonable“ simulation of $\mathfrak{F}_1$.

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A problem can arise if the state q_n appears in the initial configuration. Since it does not belong to the alphabet of $\mathfrak{F}_1$, it corrupts the whole evolution of the second layer.

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A problem can arise if the state q_n appears in the initial configuration. Since it does not belong to the alphabet of $\mathfrak{F}_1$, it corrupts the whole evolution of the second layer.

To avoid this, we use the fact that the counters generated by $\boxed{*}$ states have the priority over q_n.

Rewriting the initial configuration of the second layer

Take any state x_0 in the alphabet of $\mathfrak{F}_1$. The orbit of $x_0^{\mathbb{Z}}$ under the action of $\mathfrak{F}_1$ is ultimately periodic, hence it is described by a finite amount of data.

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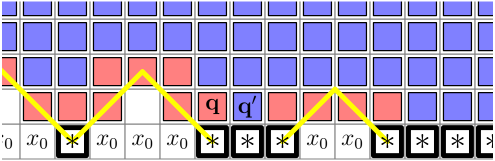
We can be sure that the second layer state is not q_n when the first layer state is $\boxed{*}$. Then we pretend that every other state of the second layer in the initial configuration is x_0.



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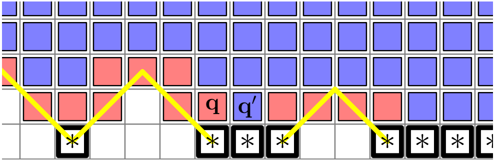

Perspectives

- A stronger theorem at level 2 or 3 of the arithmetical hierarchy.
- A proof that there exist properties at every level of the hierarchy.
- Examples of CA with a trivial generic limit set and complicated μ-limit set or the converse.