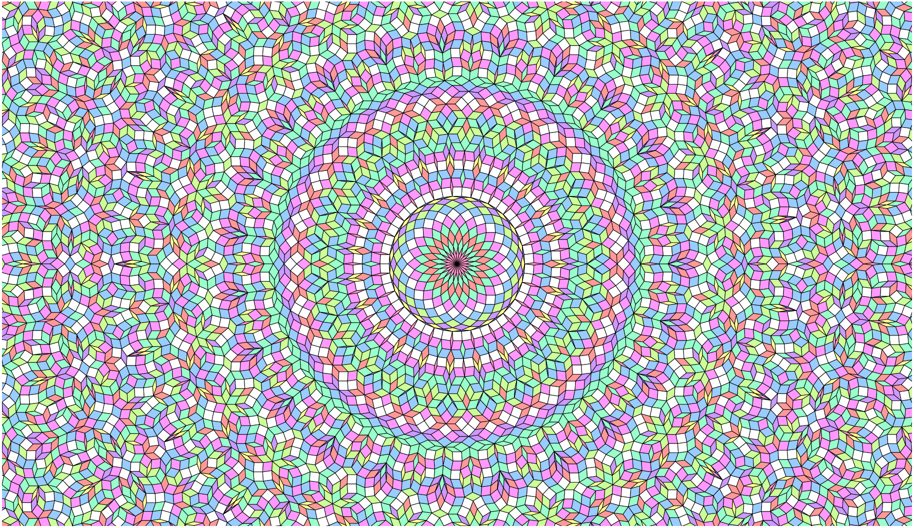
An Effective Construction for Cut-And-Project Rhombus Tilings with Global n-Fold Rotational Symmetry

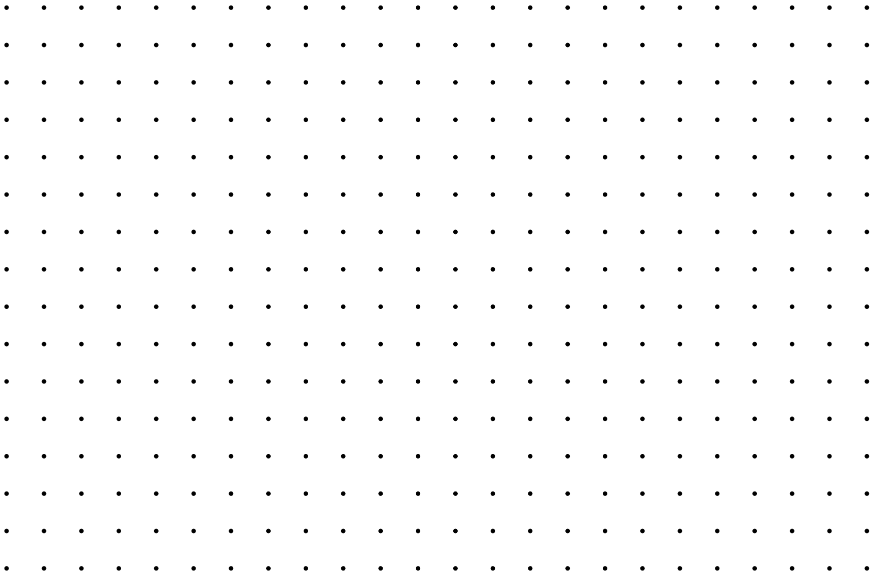
Victor H. Lutfalla

11 juillet 2021

Tiling : $P_{23}(\frac{1}{23})$



Cut



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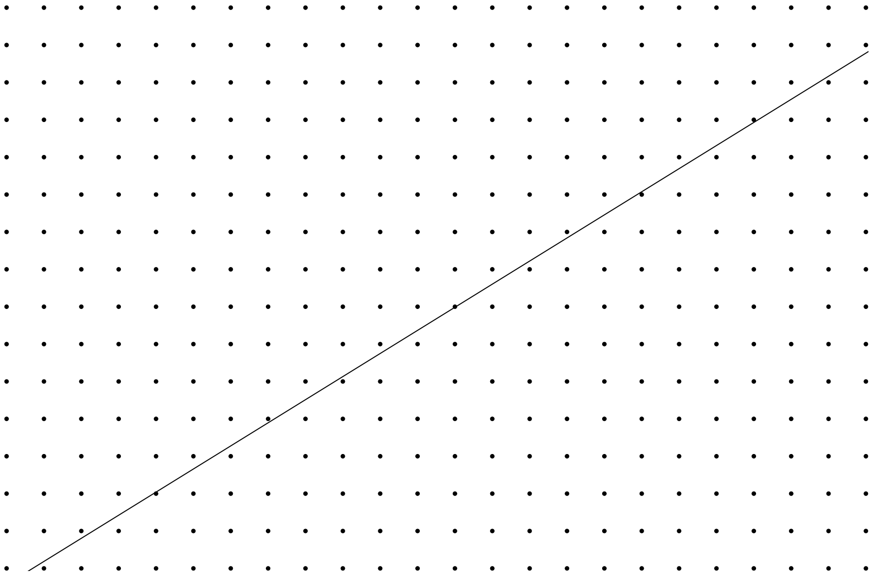

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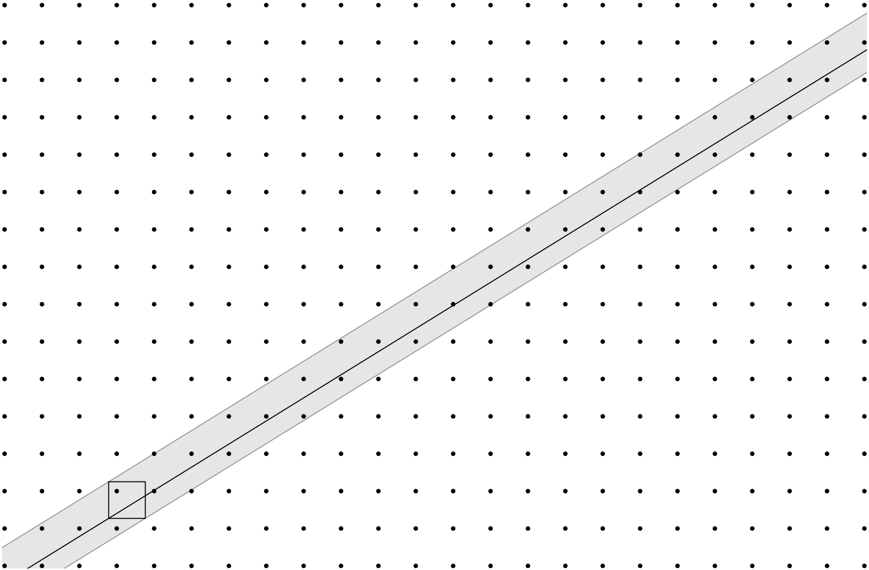

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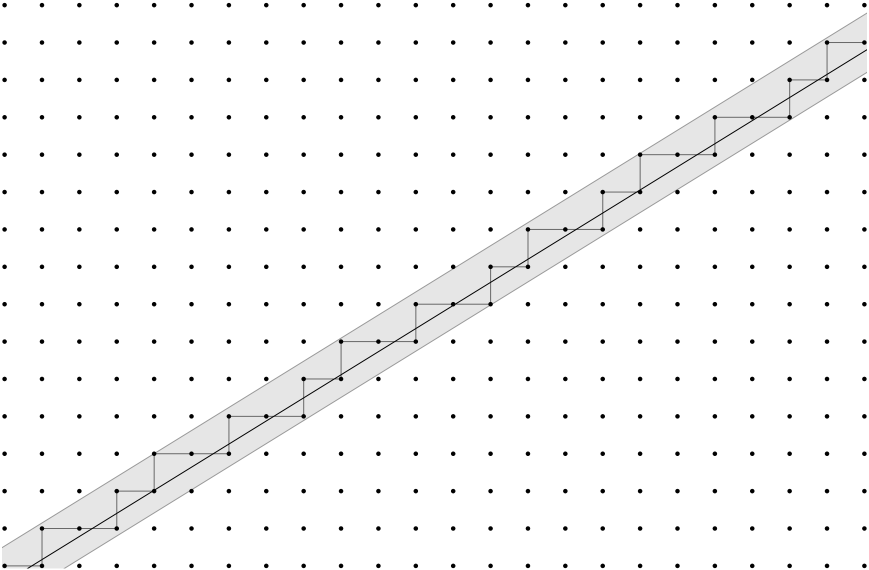
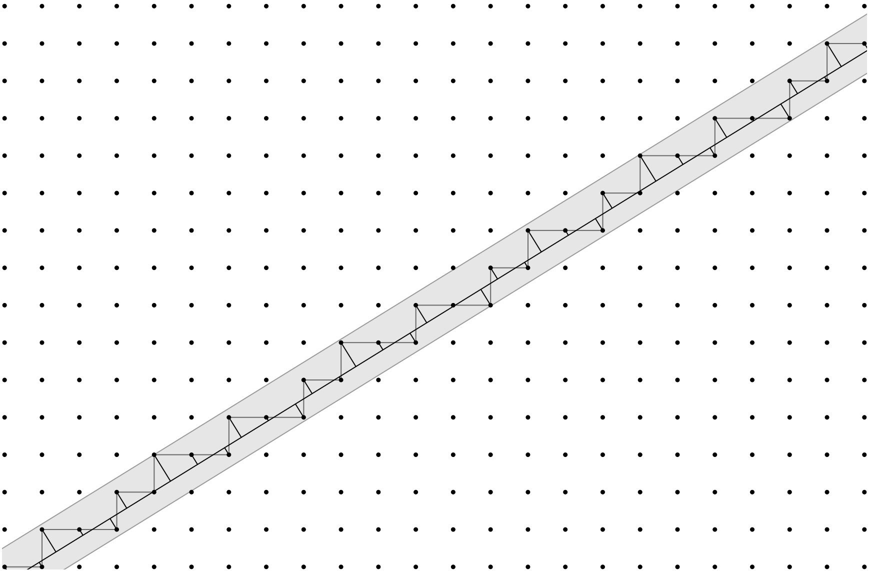


Project



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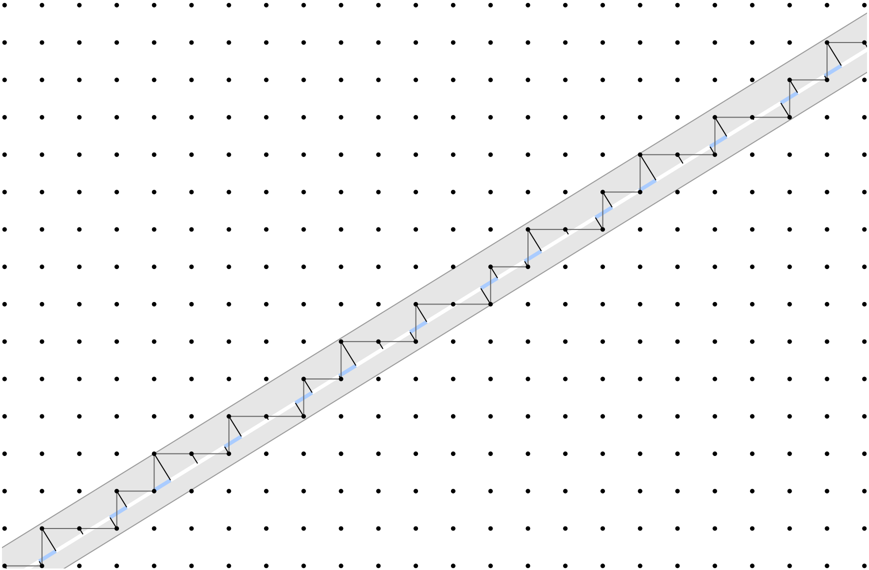
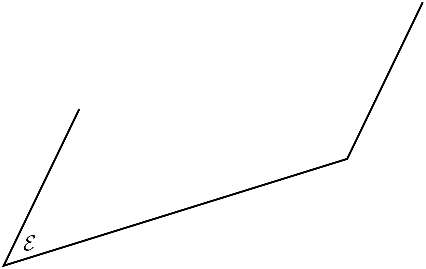


Cut-and-project tilings [Baake and Grimm, 2013]

The cut-and-project tiling of slope $\mathcal{E}$ and thickness W in $\mathbb{R}^n$ has vertex set

$$\pi_{\mathcal{E}}\left((\mathcal{E} + W) \cap \mathbb{Z}^n\right)$$



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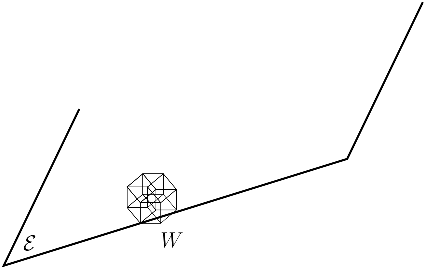

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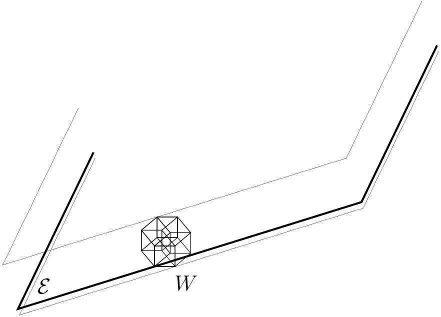
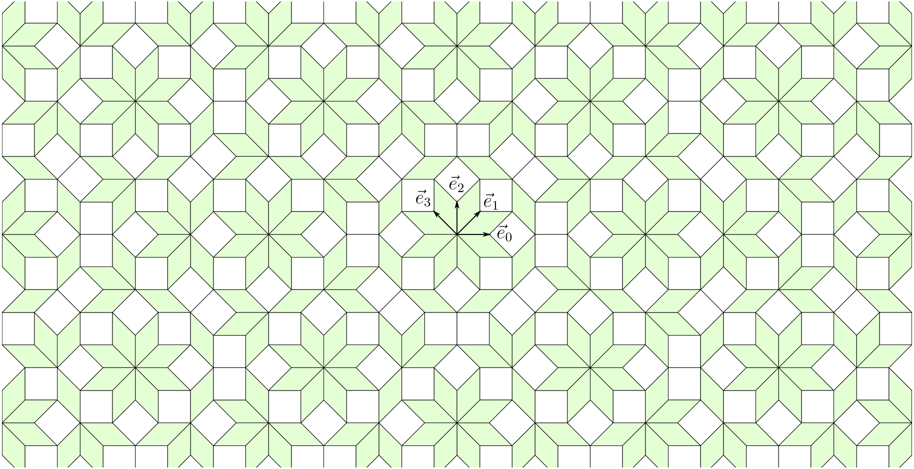


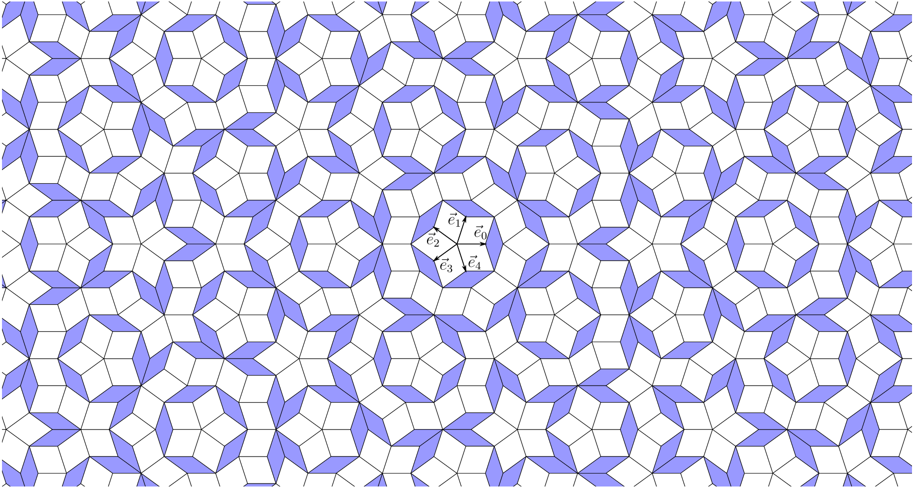
Ammann-Beenker tiling [Beenker, 1982]

Slope $\mathcal{E}_4 = \left\langle (\cos \frac{k\pi}{4})_{0 \leq k < 4}, (\sin \frac{k\pi}{4})_{0 \leq k < 4} \right\rangle$.
Thickness W = unit hypercube of $\mathbb{R}^4$.



Penrose tiling [Penrose, 1974; De Bruijn, 1981]

Slope $\mathcal{E}_5 = \left\langle (\cos \frac{2k\pi}{5})_{0 \le k < 5}, (\sin \frac{2k\pi}{5})_{0 \le k < 5} \right\rangle$.
Thickness W = unit hypercube of $\mathbb{R}^5$.



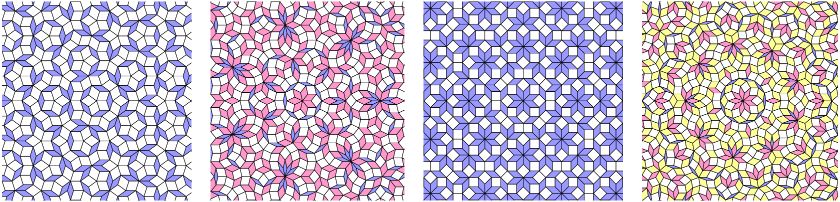
n-fold rotational symmetry

n-fold rotational symmetry: invariant by rotation of angle $\frac{2\pi}{n}$.

Theorem (Crystallographic restriction)

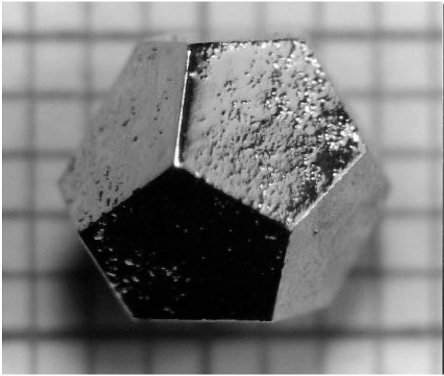
If a periodic tiling has n-fold rotational symmetry then $n \in \{1, 2, 3, 4, 6\}$.

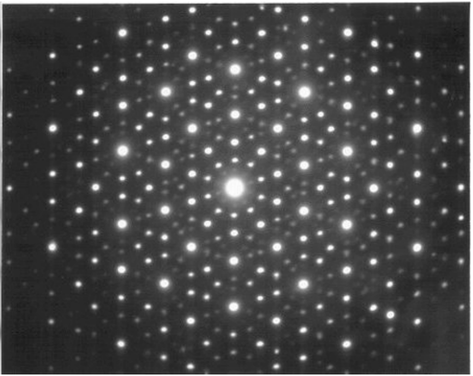
We are interested in n-fold rotational symmetry for $n \notin \{1, 2, 3, 4, 6\}$.



Quasicrystals

Cut-and-project n-fold tilings : model for quasicrystals [Senechal, 1996; Baake and Grimm, 2013].

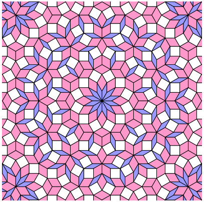


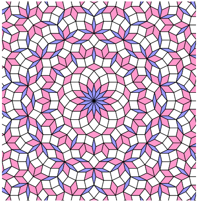


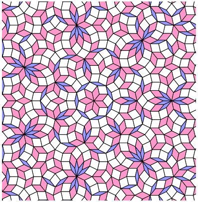
Main result

Theorem

1. *For any integer $n \geq 4$, the n-fold multigrid dual tiling $P(n)(\frac{1}{2})$ is a cut-and-project quasiperiodic edge-to-edge rhombus tiling with global $2n$-fold rotational symmetry.*
2. *For any* ***odd*** *integer $n \geq 5$, the n-fold multigrid dual tiling $P(n)(\frac{1}{n})$ is a cut-and-project quasiperiodic edge-to-edge rhombus tiling with global n-fold rotational symmetry.*





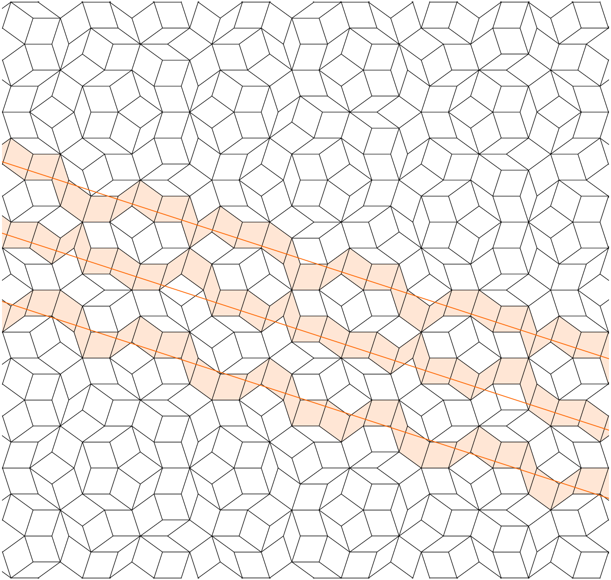


Applications

1. Gives an effective construction for Cut-and-project tilings with *n*-fold rotational symmetry.
   Software to compute these tilings :
   [Lutfalla, 2021a] doi:10.5281/zenodo.4698387.

2. These tilings are used in [Kari and Lutfalla, 2020, Lutfalla, 2021b] to define the Planar Rosa substitution discrete planes.

Chains in rhombus tilings



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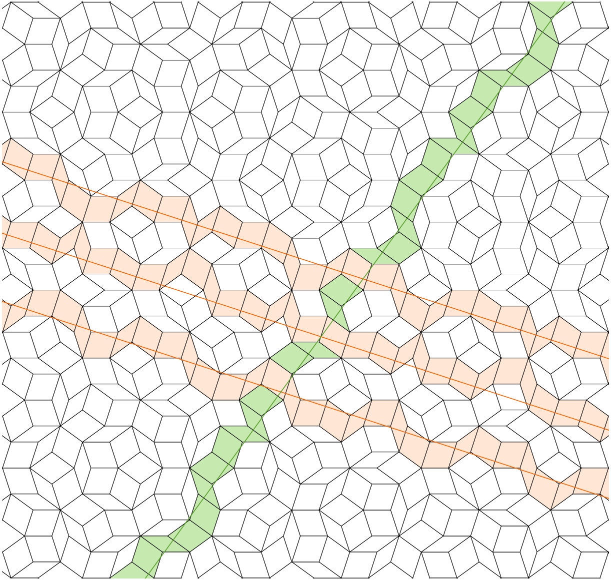


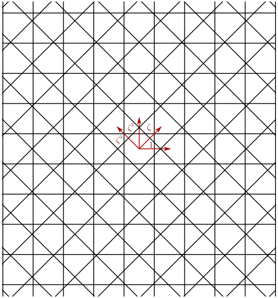
Definition : multigrid [De Bruijn, 1981]

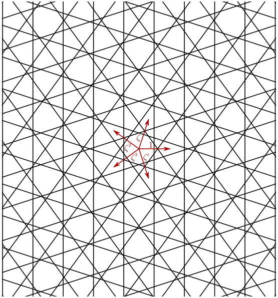
Given a complex number ζ and a real γ we define the grid $H(\zeta, \gamma)$ as:

$$H(\zeta, \gamma) := \{z | \, Re(z \cdot \overline{\zeta}) - \gamma \in \mathbb{Z}\}$$

Let $\zeta = e^{\mathrm{i}\frac{2\pi}{n}}$ if n is odd, or $\zeta = e^{\mathrm{i}\frac{\pi}{n}}$ if n is even.
Given a n-tuple of offsets γ we define the multigrid $G_n(\gamma)$ as :

$$G_n(\gamma) = \bigcup_{0 \leq k < n} H(\zeta^k, \gamma_k)$$

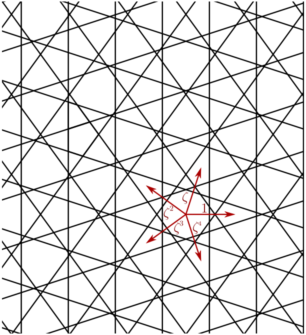


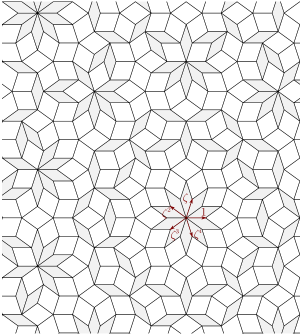


Definition : dual tiling [De Bruijn, 1981]

The multigrid dual tiling $P_n(\gamma)$ is defined by its vertex set $V(P_n(\gamma))$:

$$V(P_n(\gamma)) := f_{n,\gamma}(\mathbb{C}) \qquad \text{with} \qquad f_{n,\gamma}(z) := \sum_{k=0}^{n-1} \left\lceil Re(z \cdot \zeta^k) - \gamma_k \right\rceil \zeta^k$$





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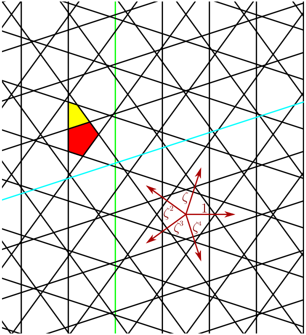
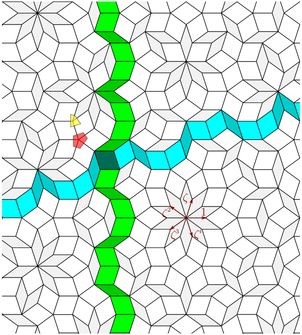


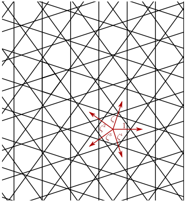


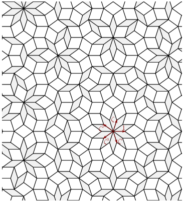
Multigrid dual tilings are cut-and-project

Theorem (Gähler and Rhyner, 1986)

Multigrid dual tilings are cut and project.

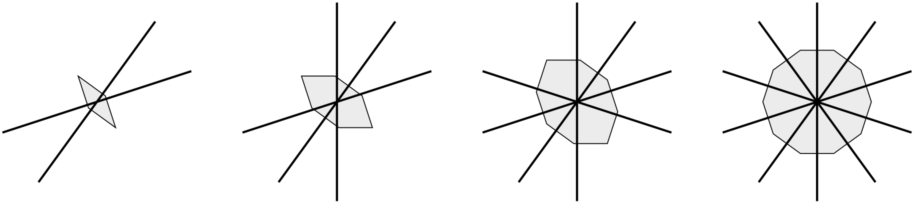
Main idea [Senechal, 1996]:
line = intersection of hyperplane $\{x \in \mathbb{R}^n | \langle x|\vec{e}_i\rangle = k\}$ with the slope $\mathcal{E}$.





Regular multigrids and rhombus tilings

Singular multigrid: there is a point where at least 3 lines intersect.
Regular multigrid: there is no such point.



Proposition

The dual of a regular multigrid is a rhombus tiling.

Goal: contruction for regular multigrids with n-fold rotational symmetry.

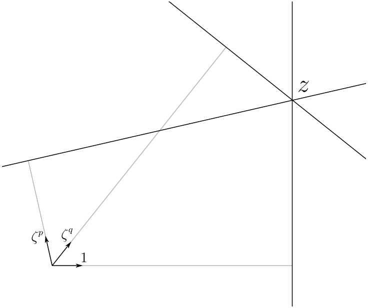
Regularity of pentagrids and cardinality considerations

The regularity of a n-fold multigrid $G_n(\gamma)$ depends on its offset γ.

1. [De Bruijn, 1981] gives a full characterization of regular pentagrids. This characterization is hard to check and is not easily generalized to n-fold multigrids.
2. Cardinality considerations prove that regular multigrids are generic. However this does not give us a construction, especially for a multigrid with global n-fold rotational symmetry.

Singularity and trigonometric equation

Singular multigrid: there is a point where at least 3 lines intersect.



$$\Leftrightarrow (Re(z) = k_0 + \gamma_0) \wedge (Re(z \cdot \zeta^{-q}) = k_q + \gamma_q) \wedge (Re(z \cdot \zeta^{-p}) = k_p + \gamma_p)$$

$$\Rightarrow (k_0 + \gamma_0) \sin \frac{2(p-q)\pi}{n} + (k_p + \gamma_p) \sin \frac{2q\pi}{n} - (k_q + \gamma_q) \sin \frac{2p\pi}{n} = 0$$

Trigonometric diophantine equation

Theorem (Conway and Jones, 1976)

The trigonometric diophantine equations with at most 4 cosine terms, one rational term and angles strictly between 0 and $\frac{\pi}{2}$ are:

$$\cos\frac{\pi}{3} = \frac{1}{2} \tag{1}$$
$$-\cos\alpha + \cos\left(\frac{\pi}{3}-\alpha\right) + \cos\left(\frac{\pi}{3}+\alpha\right) = 0 \ \left(0 < \alpha < \frac{\pi}{6}\right) \tag{2}$$
$$\cos\frac{\pi}{5} - \cos\frac{2\pi}{5} = \frac{1}{2} \tag{3}$$
$$\cos\frac{\pi}{7} - \cos\frac{2\pi}{7} + \cos\frac{3\pi}{7} = \frac{1}{2} \tag{4}$$
$$\cos\frac{\pi}{5} - \cos\frac{\pi}{15} + \cos\frac{4\pi}{15} = \frac{1}{2} \tag{5}$$
$$-\cos\frac{2\pi}{5} + \cos\frac{2\pi}{15} - \cos\frac{7\pi}{15} = \frac{1}{2} \tag{6}$$
$$\cos\frac{\pi}{7} + \cos\frac{3\pi}{7} - \cos\frac{\pi}{21} + \cos\frac{8\pi}{21} = \frac{1}{2} \tag{7}$$
$$\cos\frac{\pi}{7} - \cos\frac{2\pi}{7} + \cos\frac{2\pi}{21} - \cos\frac{5\pi}{21} = \frac{1}{2} \tag{8}$$
$$-\cos\frac{2\pi}{7} + \cos\frac{3\pi}{7} + \cos\frac{4\pi}{21} + \cos\frac{10\pi}{21} = \frac{1}{2} \tag{9}$$
$$-\cos\frac{\pi}{15} + \cos\frac{2\pi}{15} + \cos\frac{4\pi}{15} - \cos\frac{7\pi}{15} = \frac{1}{2} \tag{10}$$

Corollaries

Corollary

If $A\cos(a) + B\cos(b) = C$ *then either*

$$a = \frac{\pi}{5},\ b = \frac{2\pi}{5},\ A = -B = 2C$$

or

$$a = b = \frac{\pi}{3},\ A + B = 2C$$

Corollary

If $A\cos(a) + B\cos(b) + C\cos(c) = 0$ *then either:*

$$a = \frac{\pi}{5},\ b = \frac{\pi}{3},\ c = \frac{2\pi}{5},\ B = C = -A$$

or

$$0 < a < \frac{\pi}{6},\ b = \frac{\pi}{3} - a,\ c = \frac{\pi}{3} + a,\ B = C = -A$$

Multigrids with rational offsets for odd n

Let $n \in 2\mathbb{N}+1$ and let $\gamma = (\gamma_k)_{0\leq k<n} \in (\mathbb{Q}\cap]0,1[)^n$.

$G_n(\gamma)$ singular

$$\Rightarrow (k_0+\gamma_0)\sin\frac{2(p-q)\pi}{n} + (k_p+\gamma_p)\sin\frac{2q\pi}{n} - (k_q+\gamma_q)\sin\frac{2p\pi}{n} = 0$$

$$\Rightarrow r_0\cos\theta_0 + r_p\cos\theta_q + r_q\cos\theta_p = 0$$

$$\begin{aligned}\text{With } r_0 &:= \epsilon(\tfrac{2(p-q)\pi}{n})(k_0+\gamma_0), & \theta_0 &:= \varphi(\tfrac{2(p-q)\pi}{n})\\ r_p &:= \epsilon(\tfrac{2q\pi}{n})(k_p+\gamma_p), & \theta_q &:= \varphi(\tfrac{2q\pi}{n})\\ r_q &:= \epsilon(\tfrac{2p\pi}{n})(k_q+\gamma_q), & \theta_p &:= \varphi(\tfrac{2p\pi}{n})\end{aligned}$$

With functions $\epsilon(x) := (-1)^{\lfloor \frac{x}{\pi}\rfloor}$, $\varphi(x) := (-1)^{\lfloor \frac{2x}{\pi}\rfloor}\left(\lfloor \frac{x}{\pi}\rfloor\pi + \frac{\pi}{2} - x\right)$

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r_q &:= \epsilon\left(\tfrac{2p\pi}{n}\right)(k_q+\gamma_q), & \theta_p &:= \varphi\left(\tfrac{2p\pi}{n}\right)
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In particular $r_0, r_p, r_q \in \mathbb{Q}\setminus\{0\}$ and $\theta_0, \theta_q, \theta_p \in \pi\mathbb{Q}\cap]0,\frac{\pi}{2}[$

$\Rightarrow$ the Conway-Jones theorem and its corollaries apply.

Applying Conway-Jones for odd n

First case: suppose $\{\theta_0, \theta_p, \theta_q\} = \{\frac{\pi}{5}, \frac{\pi}{3}, \frac{2\pi}{5}\}$.
From this we get $\{\frac{2(p-q)\pi}{n}, \frac{2p\pi}{n}, \frac{2q\pi}{n}\} = \{\theta_1, \theta_2, \theta_3\}$ with

$$\theta_1 \in \varphi^{-1}(\tfrac{\pi}{5}) = \{\tfrac{3\pi}{10}, \tfrac{7\pi}{10}, \tfrac{13\pi}{10}, \tfrac{17\pi}{10}\}$$
$$\theta_2 \in \varphi^{-1}(\tfrac{\pi}{3}) = \{\tfrac{\pi}{6}, \tfrac{5\pi}{6}, \tfrac{7\pi}{6}, \tfrac{11\pi}{6}\}$$
$$\theta_3 \in \varphi^{-1}(\tfrac{2\pi}{5}) = \{\tfrac{\pi}{10}, \tfrac{9\pi}{10}, \tfrac{11\pi}{10}, \tfrac{19\pi}{10}\}$$

We have $\frac{2(p-q)\pi}{n} + \frac{2q\pi}{n} = \frac{2p\pi}{n}$ but we have no such $\theta_1, \theta_2, \theta_3$.
$\Rightarrow$ contradiction $\Rightarrow \{\theta_0, \theta_p, \theta_q\} \neq \{\frac{\pi}{5}, \frac{\pi}{3}, \frac{2\pi}{5}\}$.

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Second case: $\{\theta_0, \theta_p, \theta_q\} = \{\alpha, \frac{\pi}{3} - \alpha, \frac{\pi}{3} + \alpha\}$ for some $0 < \alpha < \frac{\pi}{6}$
With the same proof we get a contradiction.

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With the same proof we get a contradiction.

Conclusion: $r_0 \cos\theta_0 + r_p \cos\theta_q + r_q \cos\theta_p \neq 0$
For all odd n and all tuple of non-zero rational offsets γ, $G_n(\gamma)$ is regular.

Key result on the regularity of *n*-fold multigrids

Theorem (Regularity of *n*-fold multigrids)

1. *For any* ***odd*** $n \geq 5$*, and any tuple of non-zero rationals* γ*, the n-fold multigrid* $G_n(\gamma)$ *is regular.*
2. *For any* $n \geq 4$ *and any non-zero rational* γ_0 *the n-fold multigrid* $G_n(\gamma_0)$ *is regular.*

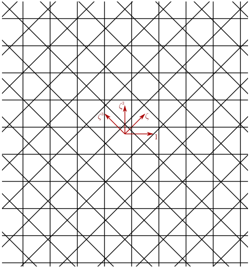
n-fold multigrid dual tilings for odd n

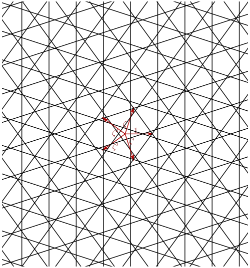
Proposition

For any n the n-fold multigrid $G_n(\frac{1}{n})$ is regular.

Proposition

For any ***odd*** *n the n-fold multigrid dual tiling $P_n(\frac{1}{n})$ is a cut-and-project rhombus tiling with n-fold rotational symmetry.*





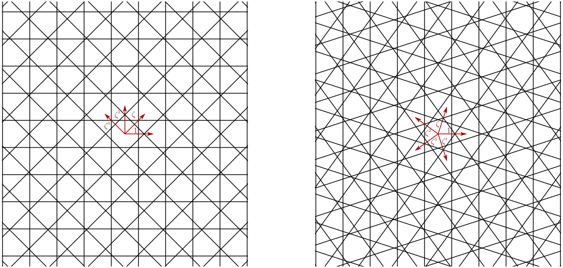
$2n$-fold multigrid dual tiling for any n

Proposition

For any n the n-fold multigrid $G_n(\frac{1}{2})$ is regular.

Proposition

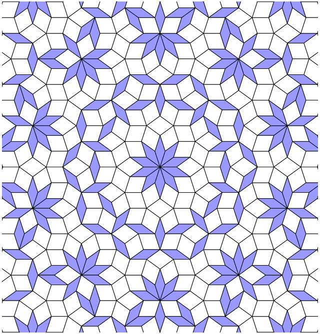
For any n the n-fold multigrid dual tiling $P_n(\frac{1}{2})$ is a cut-and-project rhombus tiling with $2n$-fold rotational symmetry.

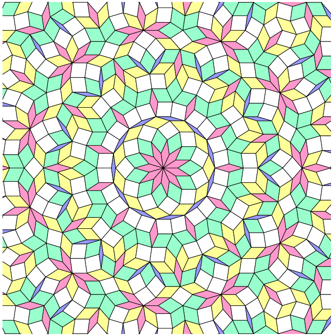


n-fold multigrid dual tiling for any n

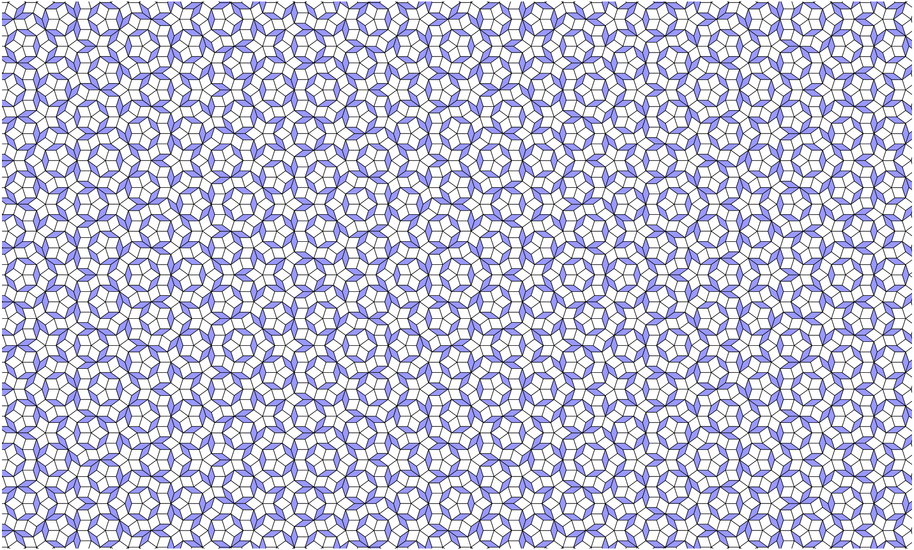
Theorem

- *If n is odd then $P_n(\frac{1}{n})$ is a cut-and-project rhombus tiling with n-fold rotational symmetry.*
- *If n is even then $P_{\frac{n}{2}}(\frac{1}{2})$ is a cut-and-project rhombus tiling with n-fold rotational symmetry.*

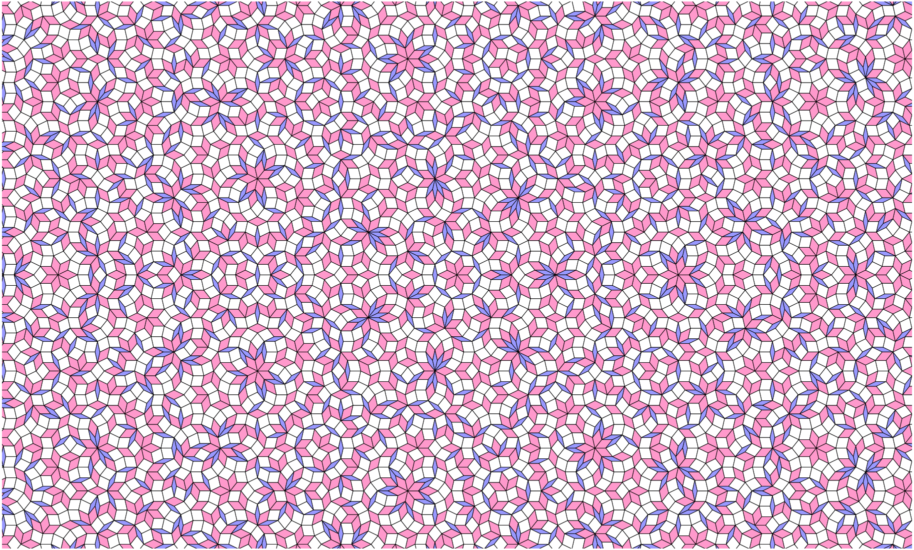




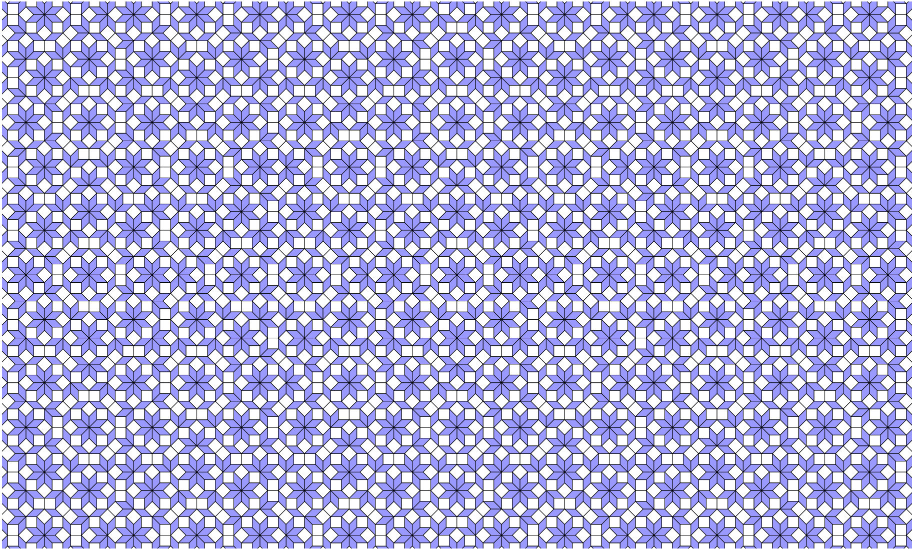
5-fold: $P_5(\frac{1}{5})$



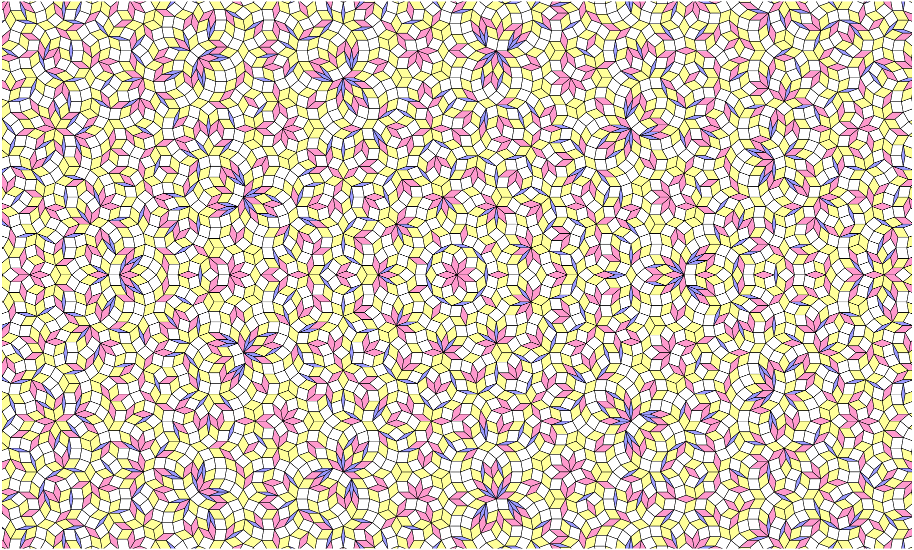
7-fold: $P_7(\frac{1}{7})$



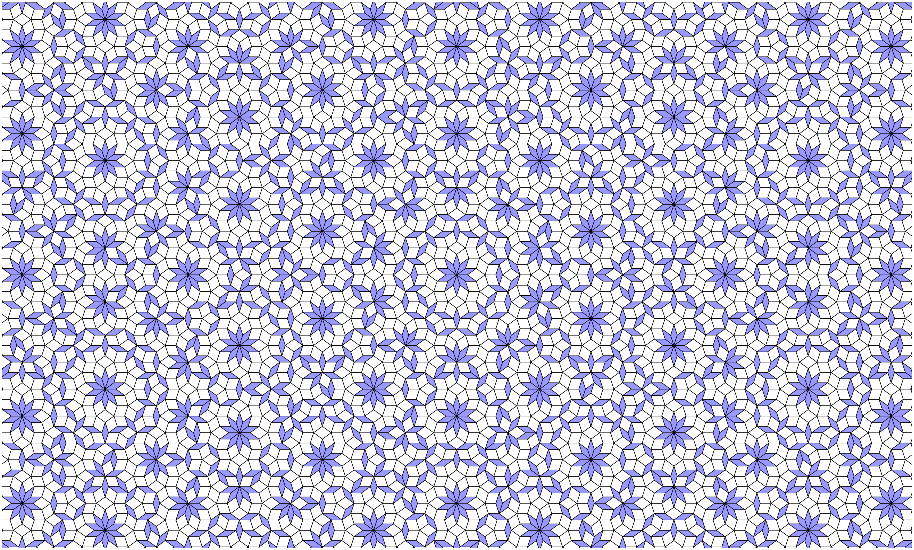
8-fold: $P_4(\frac{1}{2})$



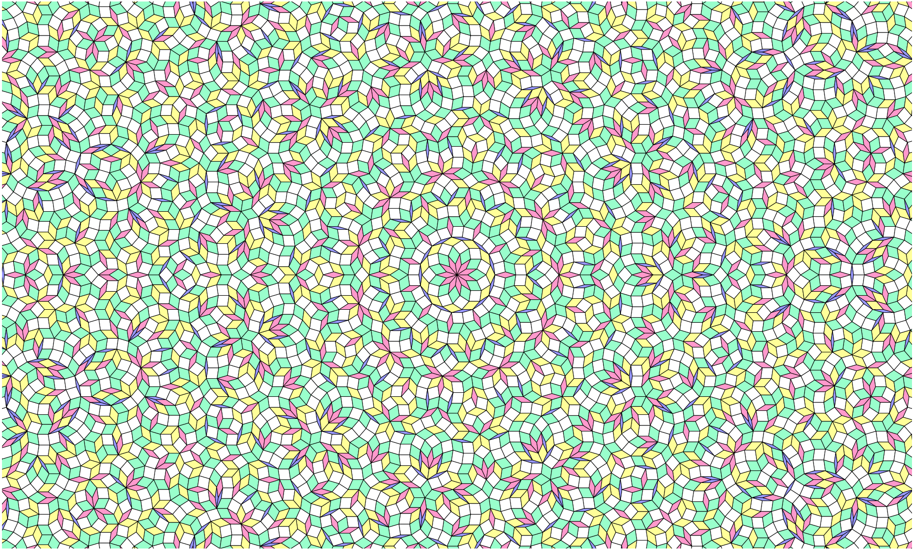
9-fold: $P_9(\frac{1}{9})$



10-fold: $P_5(\frac{1}{2})$



11-fold : $P_{11}(\frac{1}{11})$



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