Diameter mean equicontinuity and cellular automata

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- We say (X, T) is a **topological dynamical system (TDS)** if X is a compact metric space (with metric d) and $T : X \to X$ is a continuous function.
- We set $B_\delta(x) = \{y \in X : d(x, y) < \delta\}$, and the diameter of a subset A with $diam(A)$.

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- (X, T) is **equicontinuous** if $EQ = X$.
- (X, T) is **almost equicontinuous** if EQ is a dense set.

Proposition(Kůrka)

Let $(A^{\mathbb{Z}}, T)$ be a CA. The following conditions are equivalent.

(1) $(A^{\mathbb{Z}}, T)$ is equicontinuous.

(2) There exists a preperiod $m \geq 0$ and a period $p > 0$, such that $T^{m+p} = T^m$.

Proposition(Kůrka)

Let (X, T) be a CA. If $x \in X$ an equicontinuity point then x is locally eventually periodic, i.e., for every $i \in \mathbb{Z}$ we have that $T^n x_i$ is an eventually periodic sequence (of n).

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- We say x is a **diam-mean equicontinuity point** if for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\limsup_{n\to\infty} \frac{\sum_{i=0}^{n} \mathrm{diam}(T^i B_\delta(x))}{n+1} < \varepsilon.$$

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- Mean equicontinuity can be defined using the Besicovitch pseudometric with respect to T.
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  Are they different for CA?

Pacman CA

In a previous paper we constructed the Pacman CA and we proved it is almost mean equicontinuous. Let $A = \{$, , , , , $\}$. This CA has left and right radius 1.

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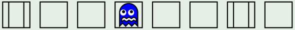
-  empty space,
-  empty door,
-  pacman,
-  ghost.
-  keymaster ghost, and
-  door with ghost.

Example

Let $w =$ . We show a section of the orbit of $x = {}^{\infty}\square.w\square^{\infty}$.

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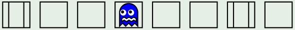
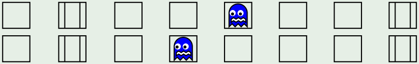
Let $w =$ . We show a section of the orbit of $x = {}^{\infty}\square.w\square^{\infty}$.



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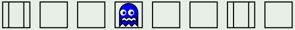
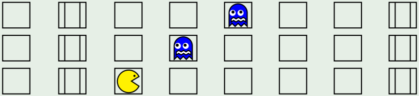
Let $w =$ . We show a section of the orbit of $x = {}^{\infty}\square.w\square^{\infty}$.



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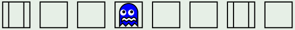
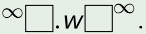
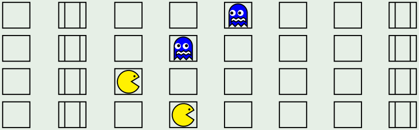
Let $w =$ . We show a section of the orbit of $x =$ .



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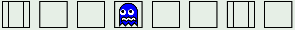
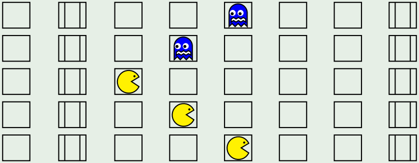
Let $w =$ . We show a section of the orbit of $x = {}^{\infty}\square.w\square^{\infty}$.



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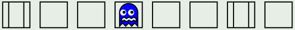
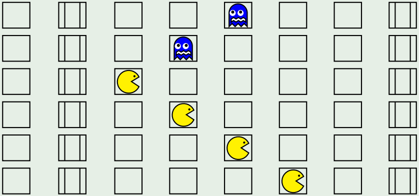
Let $w =$ . We show a section of the orbit of $x = {}^{\infty}\square.w\square^{\infty}$.



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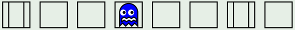
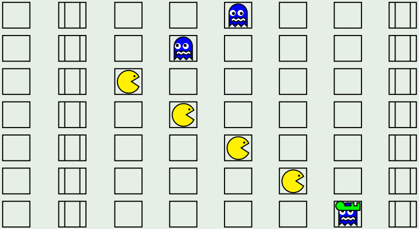
Let $w =$ . We show a section of the orbit of $x = {}^{\infty}\square.w\square^{\infty}$.



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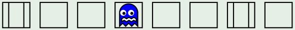
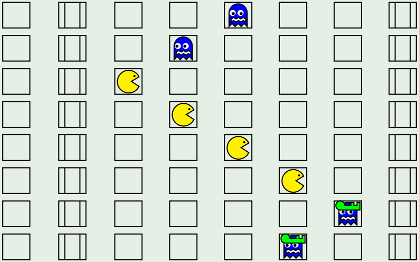
Let $w =$ . We show a section of the orbit of $x = {}^{\infty}\square.w\square^{\infty}$.



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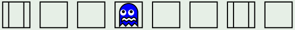
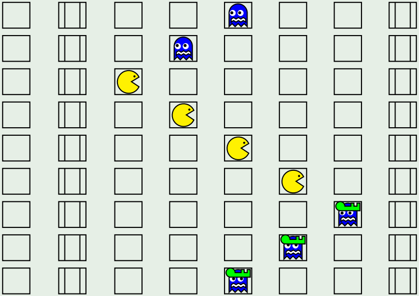
Let $w =$ . We show a section of the orbit of $x = {}^{\infty}\square.w\square^{\infty}$.



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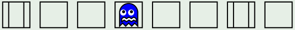
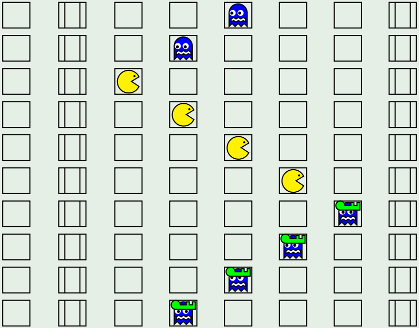
Let $w =$ . We show a section of the orbit of $x = {}^{\infty}\square.w\square^{\infty}$.



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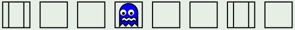
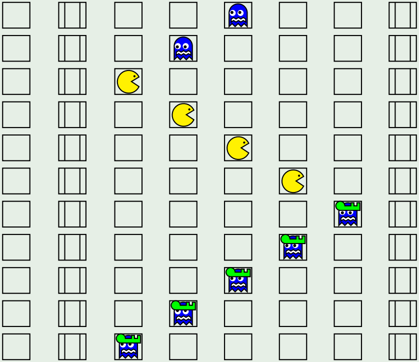
Let $w =$ . We show a section of the orbit of $x = {}^{\infty}\square.w\square^{\infty}$.



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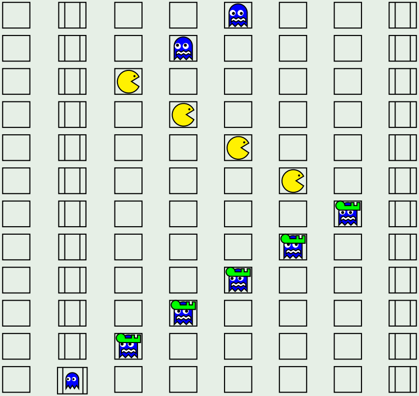

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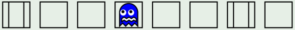
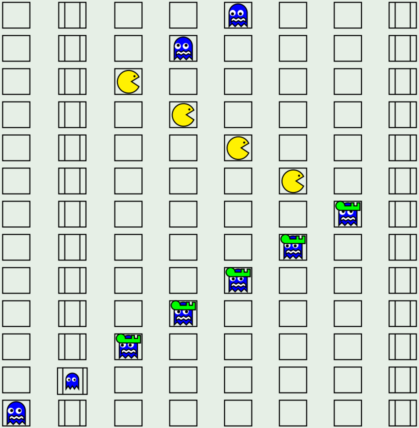
Let $w =$ . We show a section of the orbit of $x = {}^{\infty}\square.w\square^{\infty}$.



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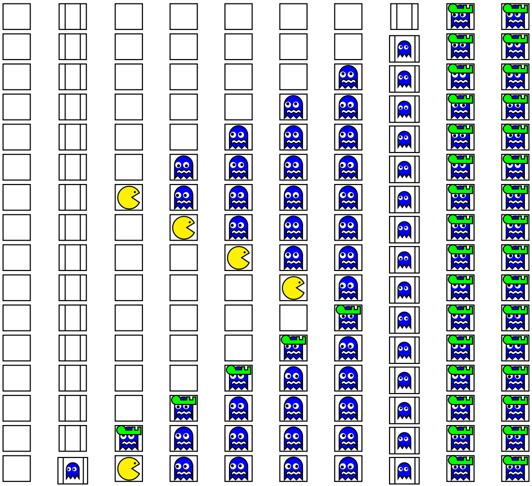

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Why not almost diam-mean equicontinuous?

For every open set U there exists $N > 0$ such that for every $n > N$ there exists $x, y \in U$, such that $(T^n x)_0 =$  and $(T^n y)_0 = \square$.

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Definition

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Theorem (Kůrka's dichotomy)

Let (X, T) be a CA. Then (X, T) is not sensitive if and only if it is almost equicontinuous.

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We do not know if the previous result holds for surjective CA.