Reversible cellular automata in presence of noise rapidly forget everything!

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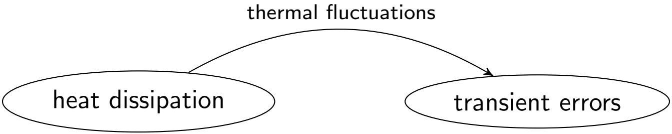
What this talk is about

- A fundamental problem with reversible computing
  $\longrightarrow$ Hypersensitivity to external noise
  - Implicitly acknowledged by Bennett, Toffoli, ...
  - Theorem: The limitation can be quantified.
  - Interesting follow-up questions
- Information-theoretic argument
  $\longrightarrow$ Evolution of entropy
  $\longrightarrow$ A bootstrap lemma

Computation with physical components

Challenges in building (very small) computers



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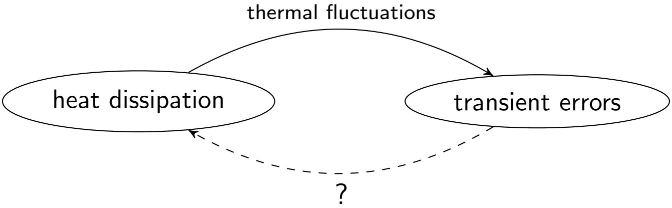

Computation with physical components

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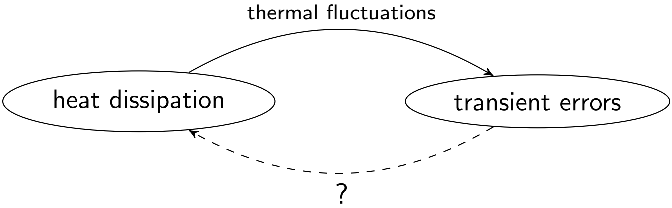


Warning

Our discussion will be limited to the classical setting!

[... although the quantum scenario is expected to be similar.]

Reversible computing

Landauer's principle (1961)

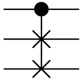
The amount of heat dissipated by erasing 1 bit of information is at least $kT \ln 2$.

Bennett (1973, 1982, 1989)

Every computation can be *efficiently* simulated by a reversible computer.

Fredkin and Toffoli (1982)

A reversible universal logic gate:



Fredkin gate

$\longrightarrow$ Google Scholar: 613,000 results (16,000 since 2021)

Reliable computing in presence of noise

Shannon (1948):

Can we do reliable communication through a noisy channel?

Von Neumann (1952):

Can we do reliable computation using noisy components?

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Solution:

- For logic circuits: Yes, if we use logarithmic redundancy!
  [Von Neumann (1956), Dobrushin and Ortyukov (1977), Pippenger (1985)]
- ...
- For cellular automata: Yes, but the known solution is very sophisticated!
  [Toom (1974, 1980), Gács and Reif (1988), Gács (1986, 2001)]

Reversible and reliable?

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Can a reversible computer be reliable in the presence of noise?

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Aharonov, Ben-Or, Impagliazzo and Nisan (1996)

The polynomial-size noisy reversible circuits* have the power of the complexity class $\mathbf{NC}^1$. [Hence, exponential redundancy is needed!]

* Noise is on the wires.

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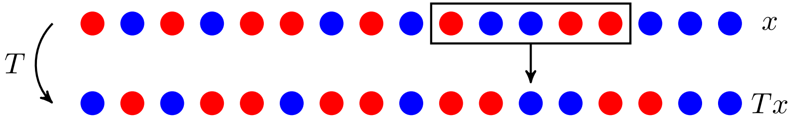
⟶ Problem: The graph of the circuit has exponential growth. [Too many wires do not fit in limited space!]

⟶ A more convenient mathematical framework to study this question is the setting of cellular automata.

Cellular automata (CA)

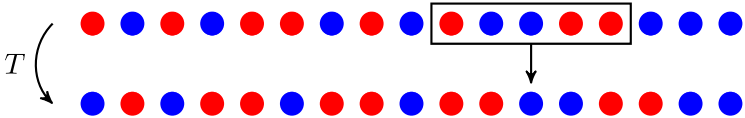


CA have "physics-like" features

- Finite number of possible states in each bounded region
- Local interactions [No action at a distance!]
- Reversibility and conservation laws can be easily implemented.
- Noise can be naturally incorporated.

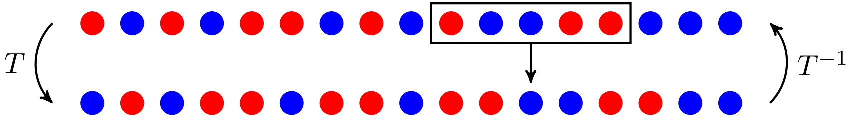
$\Longrightarrow$ Convenient for mathematical reasoning about physical implementations of computation.

Reversible CA



A CA is called reversible if

Reversible CA



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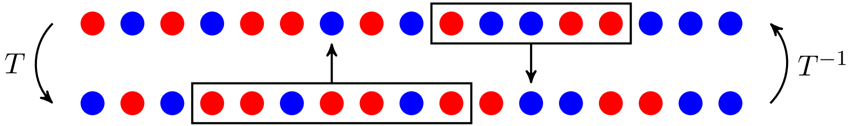
(i) T is invertible,

Reversible CA



A CA is called reversible if

(i) T is invertible,

(ii) T^{-1} is also a CA. [Redundant!]

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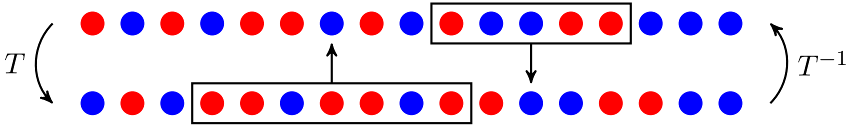

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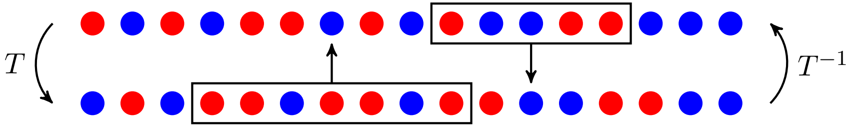
⟶ This notion of reversibility corresponds to (is more general than) the reversibility of the microscopic laws of physics.

⟶ A non-reversible CA corresponds to a system which dissipates heat. [by Landauer's principle]

Computing with reversible CA



Toffoli (1977)

Every d-dimensional CA can be simulated by a $(d+1)$-dimensional reversible CA.

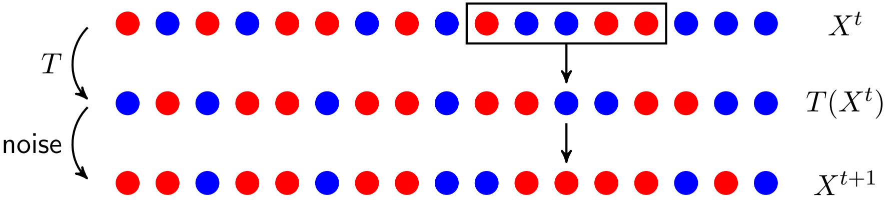
Margolus (1984)

There exists a simple computationally universal two-dimensional reversible CA (the billiard ball model).

Morita and Harao (1989), Dubacq (1995)

There exist simple and efficient computationally universal one-dimensional reversible CA.

CA + noise



Cellular automata subject to noise

At each step,

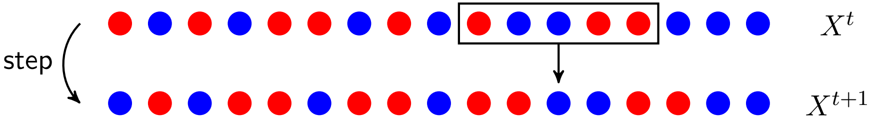
a) first, apply the deterministic CA,

b) then, add noise independently at each site.

[Various models of noise possible!]

$\rightsquigarrow$ A special type of probabilistic cellular automaton (PCA).

Probabilistic cellular automata (PCA)



PCA are similar to CA, except that

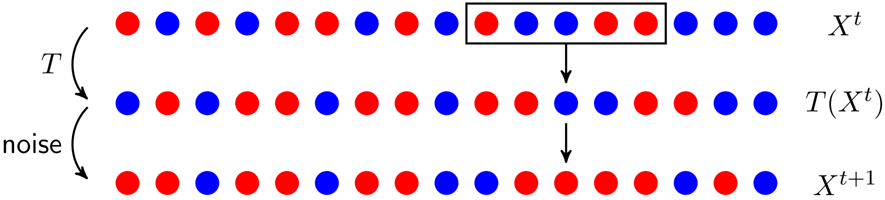
- The local rule is probabilistic! [Described by a stochastic matrix]
- Symbols at different sites are updated independently.

PCA are discrete-time Markov processes

- The state at time t is a random configuration X^t.
- The transition kernel has the Feller property.

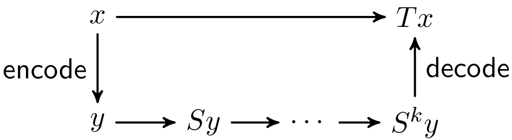
[Discrete-time variants of interacting particle systems]

Computing with noisy CA



Problem (Reliable simulation)

Can we "simulate" a CA T with another CA S that is "reliable against sufficiently weak noise"?



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Find a CA that, in presence of sufficiently weak noise is cable of "remembering" at least 1 bit of information indefinitely!

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Precise formulation in the language of Markov processes:

Problem (Ergodicity of noisy CA)

Find a CA that, in presence of sufficiently weak noise remains non-ergodic!

[Ergodicity: having a unique stationary measure that attracts every trajectory]

Computing with noisy CA

Problem (Reliable simulation)

Can we “simulate” a CA T with another CA S that is “reliable against sufficiently weak noise”?

Toom (1974, 1980)

There exists a broad family of of CA in two and higher dimensions that remain non-ergodic in presence of noise.

Gács and Reif (1988)

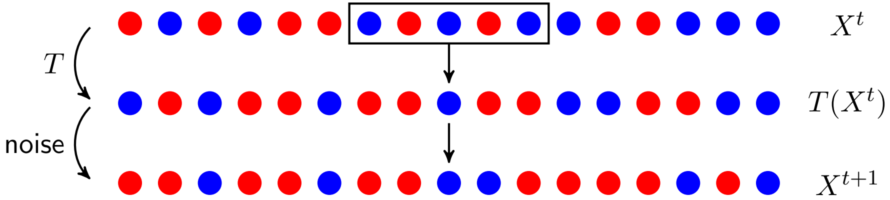
Every d-dimensional CA can be reliably simulated by a $(d+2)$-dimensional CA. [3d reliable computer not practical!]

Gács (1986, 2001)

There exists a one-dimensional intrinsically universal CA that is reliable in presence of noise!

[Very sophisticated construction with astronomical number of symbols!]

Surjective CA + additive noise



Terminology

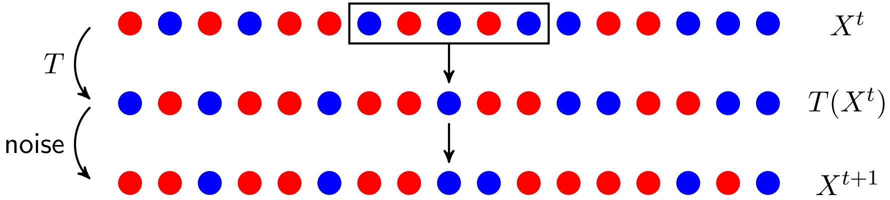
- *Surjective CA:* The global map T is onto.
- *Additive noise:* Noise adds a random value to current value, independently at each site. [modulo $|\Sigma|$]

Why care about surjective CA?

- Surjective CA include all reversible CA.

[... and have some similar properties!]

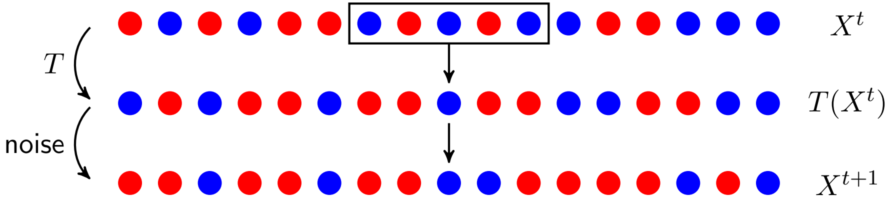
Surjective CA + additive noise



Theorem [Marcovici, Sablik, T. (2019) and T. (2021)]

Every perturbation of a surjective CA with a positive additive noise is ergodic with the uniform Bernoulli measure as its invariant measure. [Convergence is exponentially fast!]

Surjective CA + additive noise



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A reversible CA-like computer subject to noise rapidly forgets all the information in its input/software!

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- The state of any region of size n mixes in $O(\log n)$ steps.

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- A finite parallel reversible computer with n noisy components mixes in $O(\log n)$ steps. [Very limited computational power!]

[cf. Aharonov, Ben-Or, Impagliazzo, Nisan (1996)]

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Practical implication

In order to implement noise-resilient (CA-like) computers, some degree of irreversibility is necessary.

[see Bennett (1982) and Bennett and Grinstein (1985)]

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Proof idea.
Ergodicity is due to the accumulation of information.
Use entropy to measure the amount of information. □

The entropy of a discrete random variable A is

$$H(A) := -\sum_{a} \mathbb{P}(A = a) \log \mathbb{P}(A = a) \, .$$

It measures the average information content of A.

Surjective CA + additive noise

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Every perturbation of a surjective CA with a positive additive noise is ergodic with the uniform Bernoulli measure as its invariant measure. [Convergence is exponentially fast!]

Proof ingredients.

a) A surjective CA does not "erase" entropy, only "diffuses" it.

b) Additive noise increases entropy. [Sharp estimate needed!]

For each finite set of sites J and each time step $t \geq 0$, we find

$$H(X_J^t) \geq \left[1 - (1-\kappa)^t\right] |J|\, \hbar - O(|\partial J|)$$

where $\hbar := \log |\Sigma|$ is the maximum capacity of a single site.

c) A bootstrap lemma

□

Surjective CA + zero-range noise

Theorem [Marcovici, T. (2021?)]

A perturbation of a surjective CA with a positive zero-range noise is ergodic provided that both the CA and the noise preserve the same Bernoulli measure.

Proof idea.

Use pressure instead of entropy.
Use a characterization of when a surjective CA preserves a Bernoulli measure [Kari, T. (2015)]. □

The pressure of a discrete random variable A w.r.t. an energy functional f is

$$\Psi_f(A) := H(A) - \mathbb{E}[f(A)] \ .$$

It can be thought of as a contorted version of entropy.

PCA with Bernoulli invariant measure

Theorem [Marcovici, T. (2021?)]

Every positive-rate PCA that has a Bernoulli invariant measure is ergodic. [Same true for positive-rate IPS!]

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Remarks on related results

- This simultaneously extends:
  i) The above result on the ergodicity of surjective CA + noise
  ii) An earlier partial result by Vasilyev (1978)
- The entropy method goes back to Boltzmann.
  Its applications for lattice systems were pioneered by:
  $\longrightarrow$ Holley (1971), Holley and Stroock (1976) for IPS
  $\longrightarrow$ Kozlov and Vasilyev (1980) for PCA
- With the exception of Holley and Stroock (1976), the entropy method has been limited to shift-invariant starting measures.

  [Our result doesn't have this limitation.]

Entropy method for Markov processes

As a warm-up, consider the ...

Convergence theorem of Markov chains

A finite-state Markov chain is ergodic
provided that it is irreducible and aperiodic.

[Convergence is exponentially fast!]

Different proofs

- Using Perron–Frobenius theory
- Using a coupling argument
- ...
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The entropy of a discrete random variable A taking values in a finite set Σ is

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  [... for a suitable definition of conditional entropy $H(B \mid A)$]
- (continuity) $H(A)$ is continuous.

  [... as a function of the distribution of A]

Entropy method for finite-state Markov chains

Let $X^0, X^1, \ldots$ be a Markov chain with finite state space Σ and transition matrix $\theta : \Sigma \times \Sigma \to [0, 1]$.

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Proof of the convergence theorem.

We can assume $\theta > 0$.

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If $M < \log|\Sigma|$, then by compactness and continuity, we can find $A \xrightarrow{\theta} B$ with $H(A) = H(B) < \log|\Sigma|$, a contradiction. $\square$

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If $A \xrightarrow{\theta} B$, then

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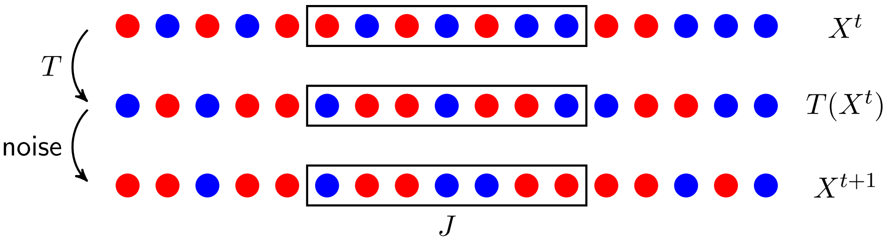
Proof of exponential convergence.

It follows from Fact II′ that

$$H(X^t) \geq \log|\Sigma| - \underbrace{(1-\kappa)^t \big[\log|\Sigma| - H(X^0)\big]}_{\to 0} \; . \qquad \square$$

Entropy method for surjective CA + additive noise



Note

- The uniform Bernoulli measure is stationary.
- In order to prove ergodicity, it is enough to show that for every finite set of sites J,

$$H(X_J^t) \to |J|\,\hbar \qquad \text{as } t \to \infty$$

where $\hbar := \log|\Sigma|$ is the maximum capacity of each site.

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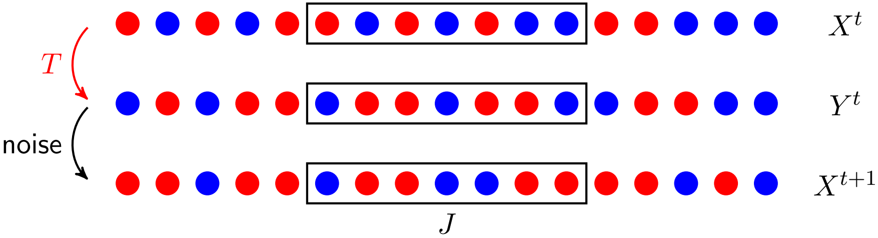

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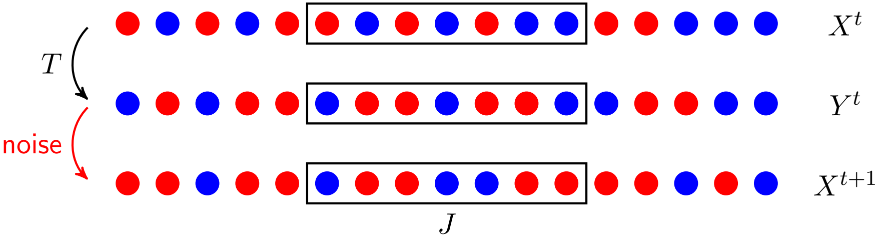


Effect of a surjective CA

A surjective CA does not "erase" entropy, only "diffuses" it:

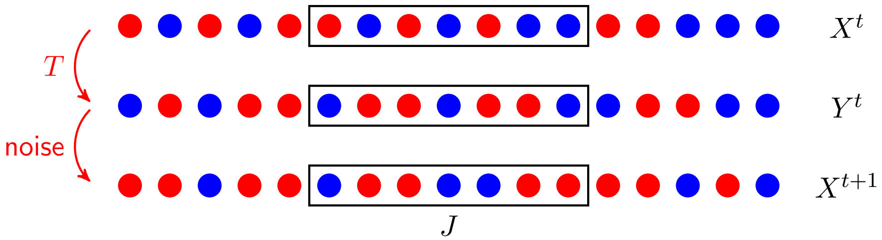
$$H(Y_J^t) \geq H(X_J^t) - O(|\partial J|)$$

Effect of additive noise

Additive noise increases entropy: $\exists$ constant $0 < \kappa \leq 1$ s.t.

$$H(X_J^{t+1}) \geq \kappa \, |J| \, \hbar + (1-\kappa) H(Y_J^t)$$

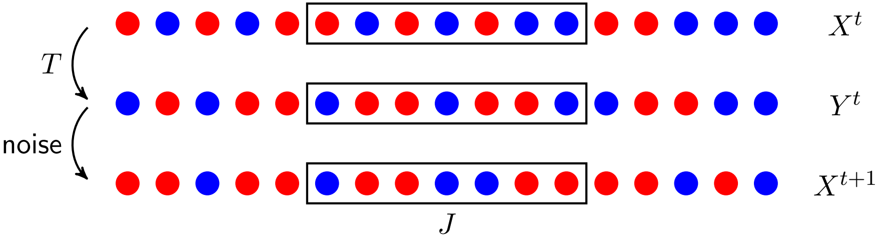
Entropy method for surjective CA + additive noise



Combined effect

$$H(X_J^{t+1}) \geq \kappa\,|J|\,\hbar + (1-\kappa)H(X_J^t) - O(|\partial J|)\ .$$

Entropy method for surjective CA + additive noise



Combined effect

$$H(X_J^{t+1}) \geq \kappa\,|J|\,\hbar + (1-\kappa)H(X_J^t) - O(|\partial J|)\ .$$

which implies

$$H(X_J^t) \geq \left[1-(1-\kappa)^t\right]|J|\,\hbar - O(|\partial J|)\ .$$

for each $t \geq 0$.

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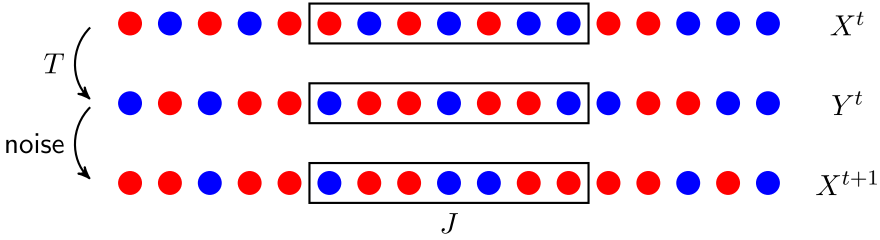

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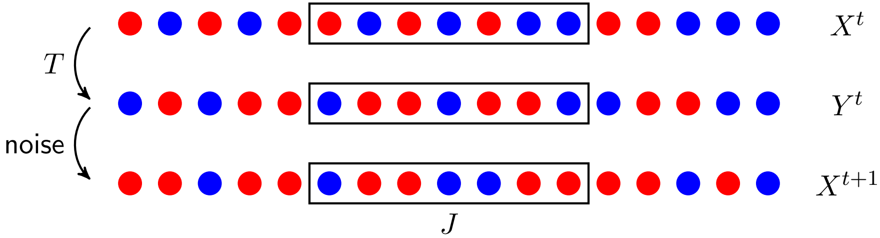


Combined effect

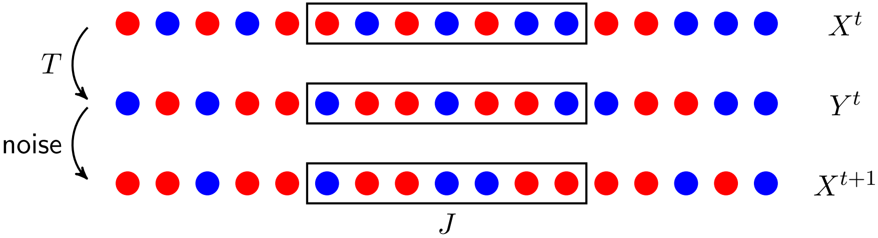
$$H(X_J^{t+1}) \geq \kappa\,|J|\,\hbar + (1-\kappa)H(X_J^t) - O(|\partial J|)\ .$$

which implies

$$H(X_J^t) \geq \underbrace{\left[1-(1-\kappa)^t\right]}_{\to 1} |J|\,\hbar - \overbrace{O(|\partial J|)}^{\text{relatively smaller}}\ .$$

for each $t \geq 0$.

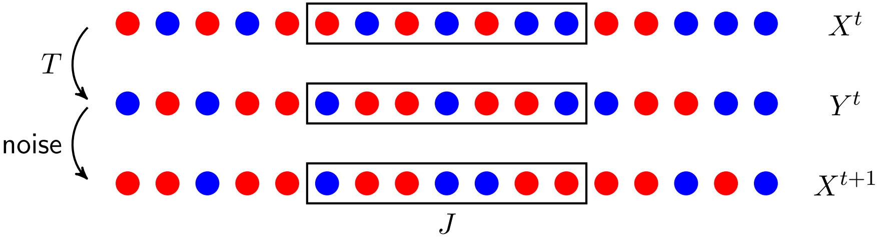
Entropy method for surjective CA + additive noise



Evolution of entropy

$$\boxed{H(X_J^t) \geq \left[1-(1-\kappa)^t\right]|J|\,\hbar - O(|\partial J|)}\,.$$

Entropy method for surjective CA + additive noise



Evolution of entropy

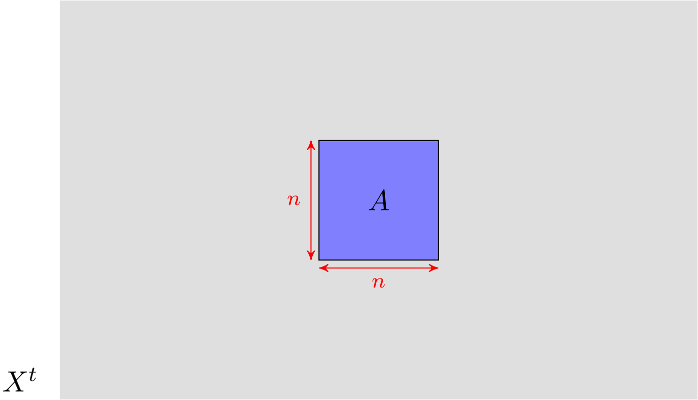
$$\boxed{H(X_J^t) \geq \left[1 - (1-\kappa)^t\right] |J|\,\hbar - O(|\partial J|)}\,.$$

In particular:

$$\underbrace{|J|\,\hbar - H(X_J^t)}_{\Xi(X_J^t)} \leq O(|\partial J|) \qquad \text{for all } t \geq a \log \frac{|J|}{O(|\partial J|)} + b$$

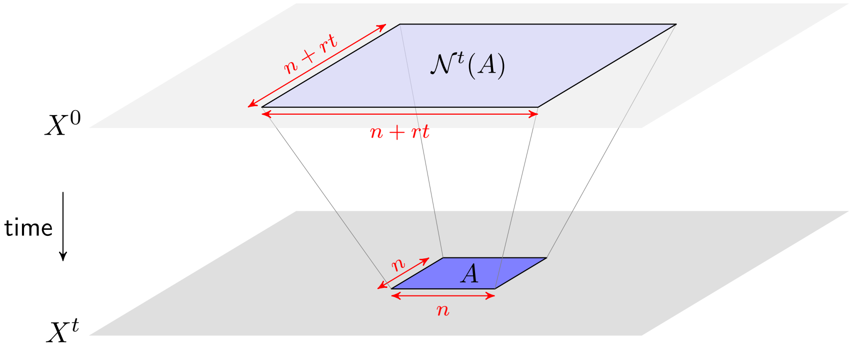
missing entropy

Bootstrapping



$$\Xi(X_A^t) \leq O(n^{d-1}) \qquad \text{for all } t \geq O(\log n)$$

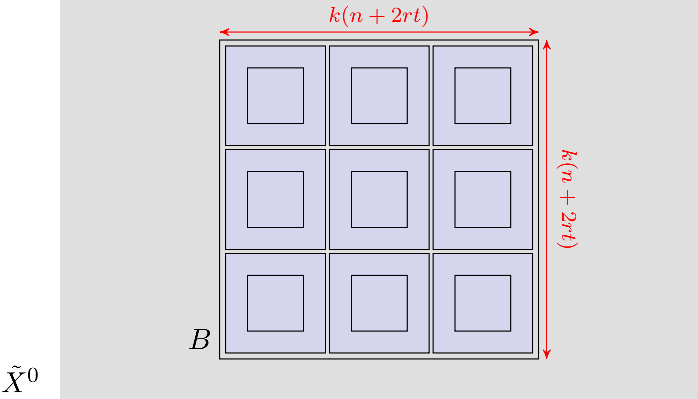
Bootstrapping



Note

The restriction of X^t to A depends only on
the restriction of X^0 to $\mathcal{N}^t(A)$,
where $\mathcal{N} = [-r, r]^d$ is the neighbourhood of the local rule.

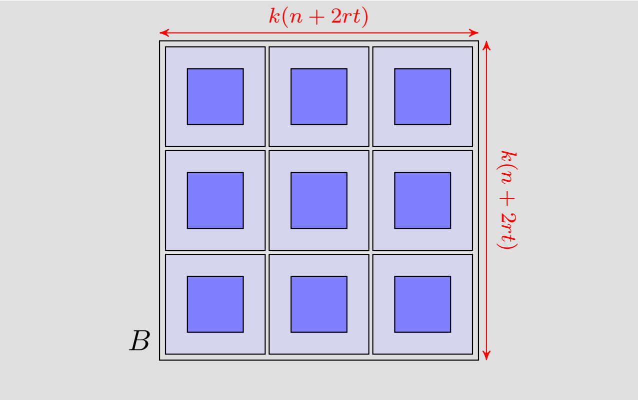
Bootstrapping



Choose $\tilde{X}^0$ such that

$$\tilde{X}^0_B \text{ contains } k^d \text{ independent copies of } X^0_{\mathcal{N}^t(A)}.$$

Bootstrapping



$\tilde{X}^t$

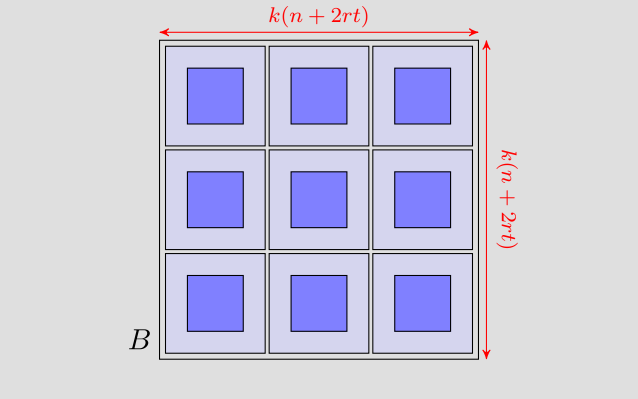
Choose $\tilde{X}^0$ such that

$$\tilde{X}^0_B \text{ contains } k^d \text{ independent copies of } X^0_{\mathcal{N}^t(A)}.$$

Then,

$$\tilde{X}^t \text{ will contain } k^d \text{ independent copies of } X^t_A \text{ inside } B.$$

Bootstrapping

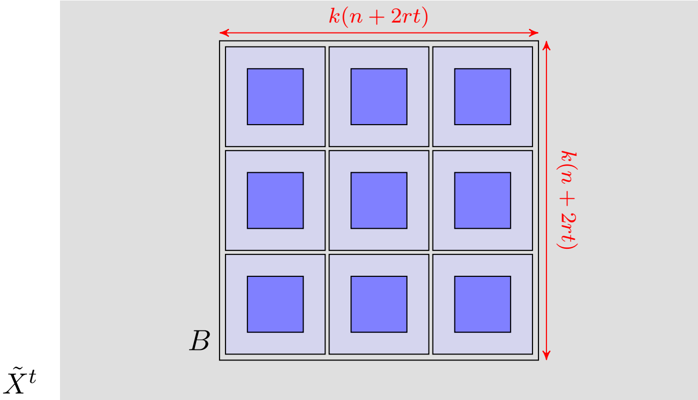


$\tilde{X}^t$

It follows that

$$k^d \, \Xi(X^t_A) \le \Xi(\tilde{X}^t_B)$$

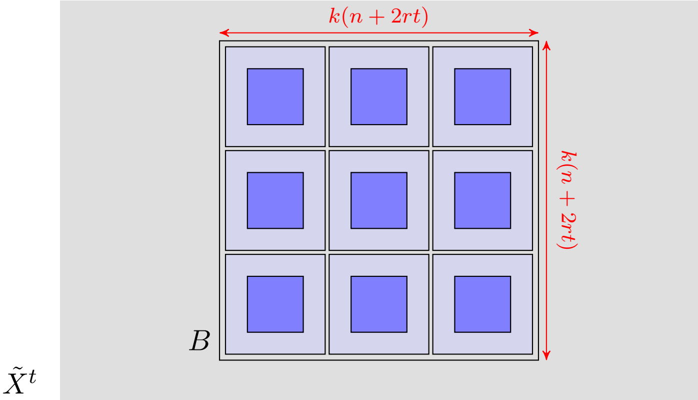
Bootstrapping



It follows that, if $t \geq O\Big(\log[k(n+2rt)]\Big)$,

$$k^d\, \Xi(X_A^t) \leq \Xi(\tilde{X}_B^t) \leq O\Big([k(n+2rt)]^{d-1}\Big)$$

Bootstrapping

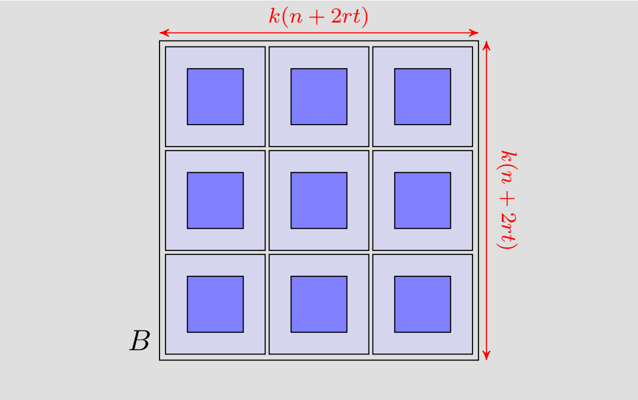


It follows that, if $t \geq O\big(\log[k(n+2rt)]\big)$,

$$k^d\,\Xi(X_A^t) \leq \Xi(\tilde{X}_B^t) \leq O\big([k(n+2rt)]^{d-1}\big)$$

Now, given $t \geq 0$, choose $k := \mathrm{e}^{ct}$ for $c > 0$ small.

Bootstrapping



$\tilde{X}^t$

Conclusion

For every $t \geq 0$ large enough,

$$\Xi(X_A^t) \leq \underbrace{O\left((n+2rt)^{d-1}\mathrm{e}^{-ct}\right)}_{\to 0} \qquad \square$$

Conclusion

Summary

A strictly reversible CA-like computer cannot be reliable in the presence of noise.

Question 1

What about a reversible TM-like computer?

Question 2

What about a quantum computer?

Question 3

How much irreversibility is needed to perform reliable computation?

Conclusion

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Thank you for your attention!