Investigations of structures in the parameter space of three-dimensional Turing-like patterns

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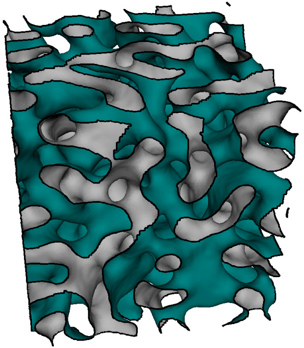
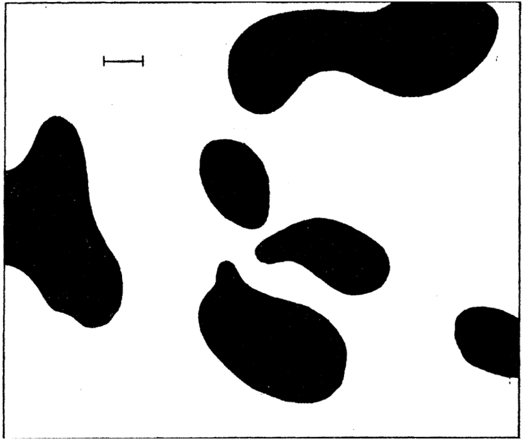
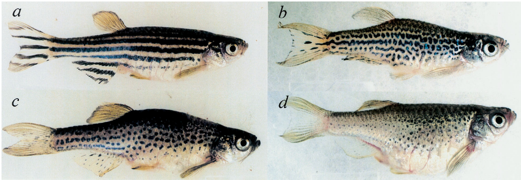
July 13, 2021



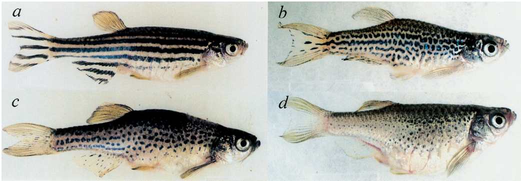
Alan M. Turing,
A theory of morphogenesis,
Philosophical Transactions B, Vol. 12,
1952.

$$\frac{dX_r}{dt} = f(X_r, Y_r) + \mu(X_{r+1} - 2X_r + X_{r-1})$$
$$\frac{dY_r}{dt} = g(X_r, Y_r) + \nu(Y_{r+1} - 2Y_r + Y_{r-1})$$





Asai, Rihito et al. *Zebrafish leopard gene as a component of the putative reaction-diffusion system*, in: Mechanisms of development, Vol. 89 (1-2), 1999.



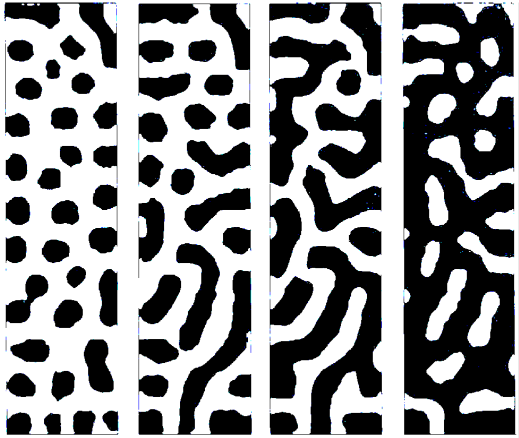
Asai, Rihito et al. *Zebrafish leopard gene as a component of the putative reaction-diffusion system*, in: Mechanisms of development, Vol. 89 (1-2), 1999.

Robert M. Pringle and Corina E. Tarnita. *Spatial self-organization of ecosystems: integrating multiple mechanisms of regular-pattern formation*, in: Annual review of Entomology, Vol. 62, 2017.

David A. Young,
A local activator-inhibitor model of vertebrate skin patterns, Mathematical Biosciences, Vol. 72(1), 1984.

$$\frac{\partial M}{\partial t} = \nabla \cdot (D \cdot \nabla M) - KM + Q$$

M	morphogen (act./inh.)
$\nabla \cdot (D \cdot \nabla M)$	diffusion
KM	chemical transformation
Q	production



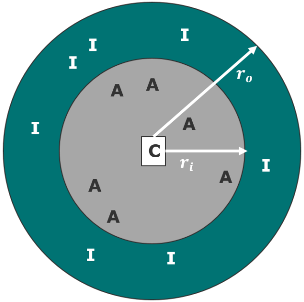
Discretization via Cellular Automata

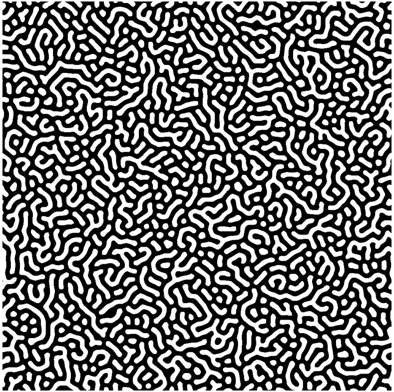
Determine status $s_{t+1}(x, y)$ by

$$s_{t+1}(x,y) = \begin{cases} 1 & \sum_{(x',y')\in B_{r_o}(x,y)} \omega_{t,(x,y)}(x',y') > 0, \\ s_t(x,y) & \sum_{(x',y')\in B_{r_o}(x,y)} \omega_{t,(x,y)}(x',y') = 0, \\ 0 & \sum_{(x',y')\in B_{r_o}(x,y)} \omega_{t,(x,y)}(x',y') < 0, \end{cases}$$

where

$$\omega_{t,(x,y)}(x',y') = \begin{cases} 0 & (x-x')^2 + (y-y')^2 > r_o^2, \\ w_1 \cdot s_t(x',y') & (x-x')^2 + (y-y')^2 < r_i^2, \\ w_2 \cdot s_t(x',y') & \text{otherwise.} \end{cases}$$

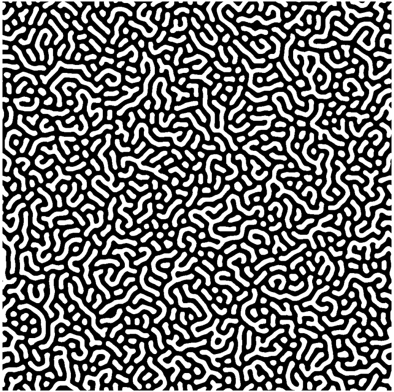




$\rho = 0.87,\ r_i = 10,\ r_o = 14$

Gary R. Greenfield. *Turing-like Patterns from Cellular Automata*, in: Proceedings of Bridges 2016, Tesselations Publishing, 2016.

Discretization via Cellular Automata



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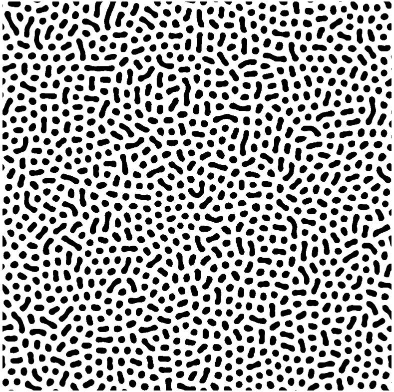


$\rho = 0.93,\ r_i = 12,\ r_o = 18.$

Gary R. Greenfield. *Turing-like Patterns from Cellular Automata*, in: Proceedings of Bridges 2016, Tesselations Publishing, 2016.

Use:

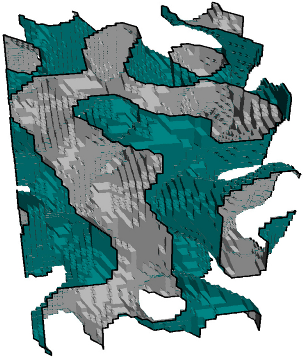
- a ball instead of the inner circle and a ball with a cavity instead of the annulus,

Generalization to 3D

Use:

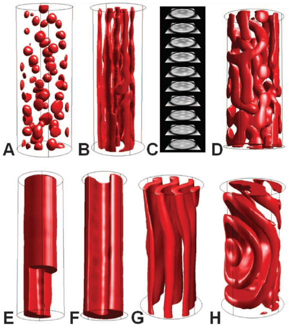
- a ball instead of the inner circle and a ball with a cavity instead of the annulus,
- marching cubes to visualize the isosurface between activated / deactivated cells.

Details:
Martin Skrodzki and Konrad Polthier. *Turing Patterns Revisited: A Peek Into The Third Dimension*, in: Proceedings of Bridges 2017, Tesselations Publishing, 2017.

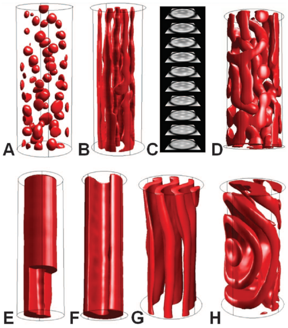


$\rho = 0.5,\ r_i = 8,\ r_o = 10.$

Generalization to 3D



Tamás Bánsági et al. *Tomography of reaction-diffusion microemulsions reveals three-dimensional Turing patterns*, in: Science, Vol. 331 (6022), 2011.



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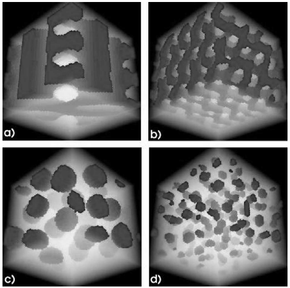
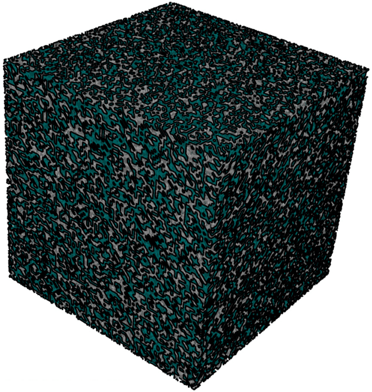
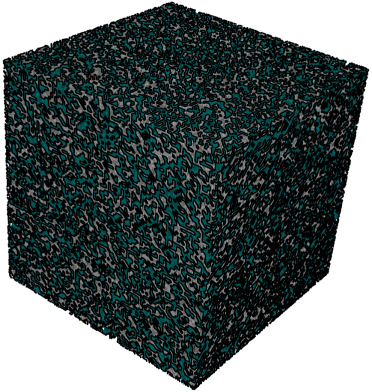


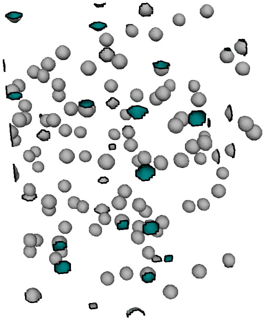
Teemu Leppänen et al. *A new dimension to Turing patterns*, in: Physica D: Nonlinear Phenomena, Vol. 168, 2002.

First 20 iterations
$\rho = 0.5,\ r_i = 4,\ r_o = 5$
Animation by
Lucas Valle Thiele



First 20 iterations
$\rho = 0.5,\ r_i = 12,\ r_o = 15$
Animation by
Lucas Valle Thiele

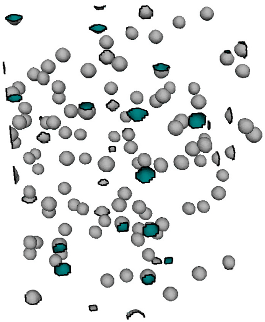




$\rho = 10^{-4},\ r_i = 4,\ r_o = 7$

Martin Skrodzki and Konrad Polthier. *Turing Patterns Revisited: A Peek Into The Third Dimension*, in: Proceedings of Bridges 2017, Tesselations Publishing, 2017.

Generalization to 3D



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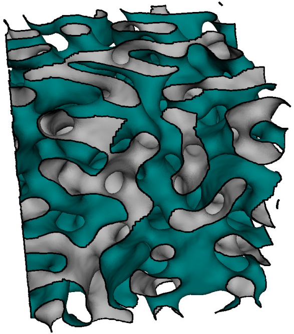
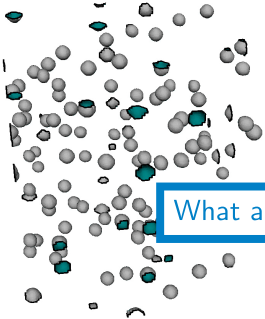
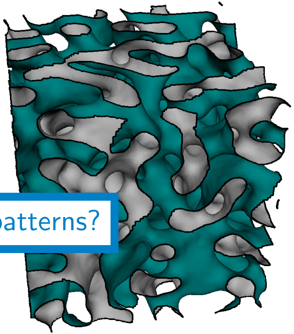


$\rho = 0.5,\ r_i = 8,\ r_o = 10$ (smoothed).

Martin Skrodzki and Konrad Polthier. *Turing Patterns Revisited: A Peek Into The Third Dimension*, in: Proceedings of Bridges 2017, Tesselations Publishing, 2017.

Generalization to 3D





What about "two-dimensional" patterns?

$\rho = 10^{-4},\ r_i = 4,\ r_o = 7$

$\rho = 0.5,\ r_i = 8,\ r_o = 10$ (smoothed).

Martin Skrodzki and Konrad Polthier. *Turing Patterns Revisited: A Peek Into The Third Dimension*, in: Proceedings of Bridges 2017, Tesselations Publishing, 2017.

Discretize the parameter space as:

- $r_i, r_o \in \{1, \ldots, 40\} \subseteq \mathbb{N}$
- $\rho = \sigma(x)$, $x \in \{0, \ldots, 120\}$

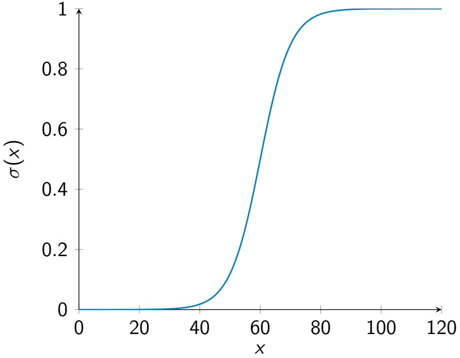
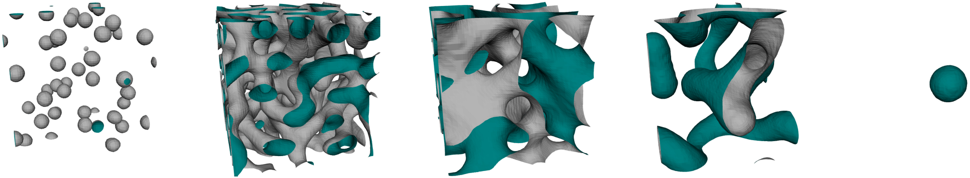


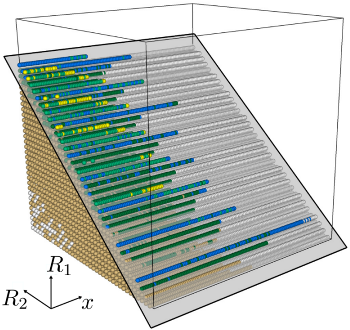
Figure: Domain $\mathcal{I} = \{0, \ldots, 120\}$ is mapped onto probability values $\sigma(x) \in [0, 1]$.

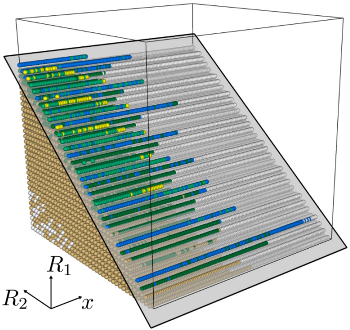
Investigating the Parameter Space



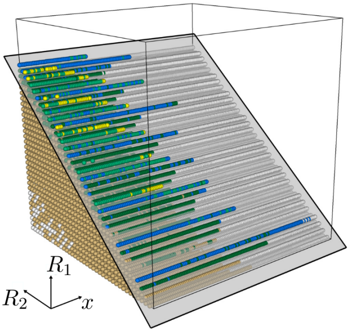
(a) DC spheres ■ “zero-dimensional” (b) DC pipes ■ “one-dimensional” (c) areas ■ “two-dimensional” (d) UC pipes ■ “one-dimensional” (e) UC spheres ■ “zero-dimensional”

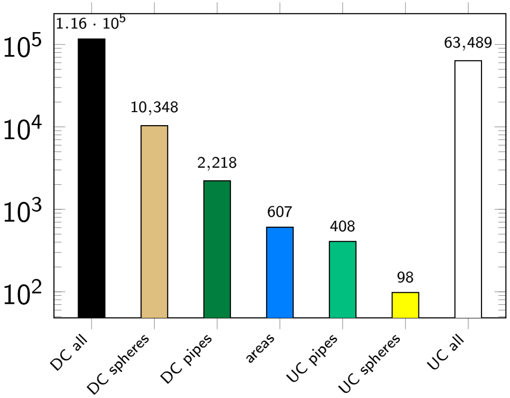
Figure: Five of the seven possible Turing-like patterns in a three-torus.


R_1
R_2
x



$$\text{vol(activator)} = \text{vol(inhibitor)} \quad \Rightarrow \quad \frac{4}{3}\pi r_i^3 = \frac{4}{3}\pi r_o^3 - \frac{4}{3}\pi r_i^3 \quad \Rightarrow \quad r_i = \frac{r_o}{\sqrt[3]{2}}.$$





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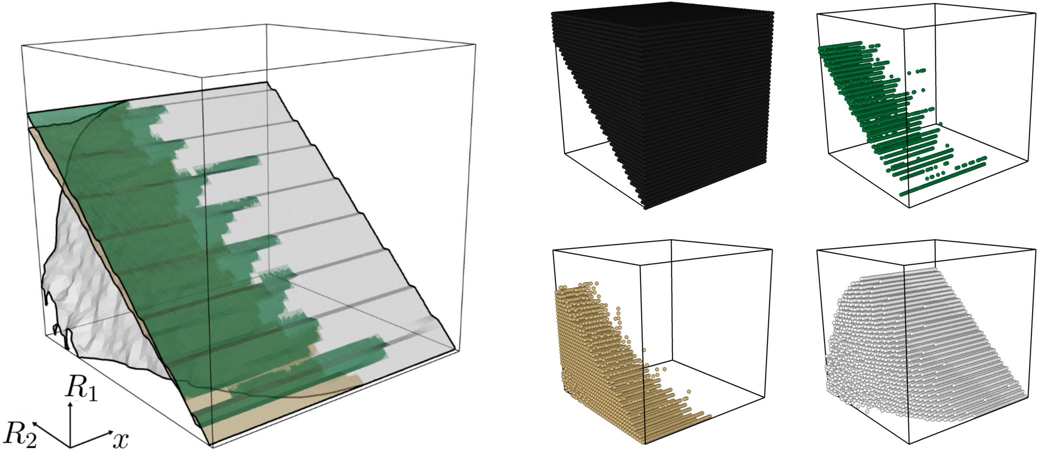
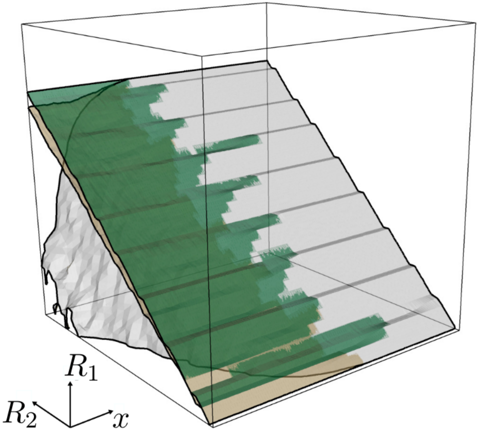
Investigating the Parameter Space

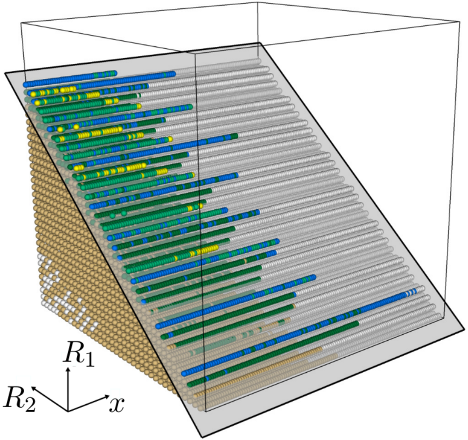




Conjecture

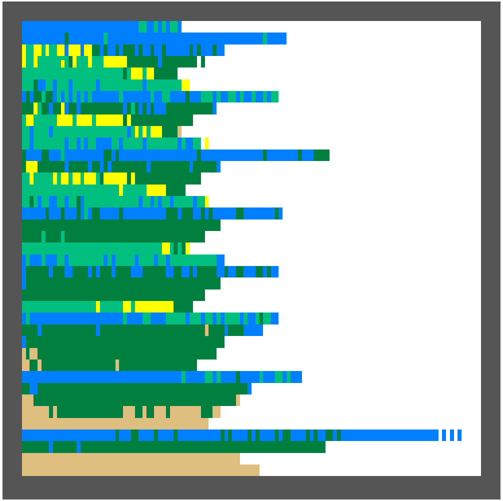
The wedge, formed in the parameter space by the "DC pipes" triples, given as volume between the beige and dark green isosurface, will grow larger and wider with increasing values for R_1 and R_2.

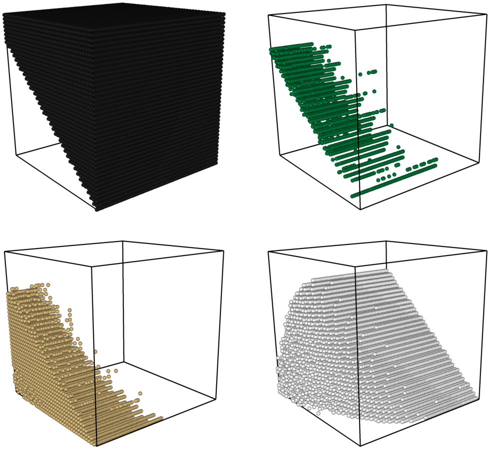
Investigating the Parameter Space



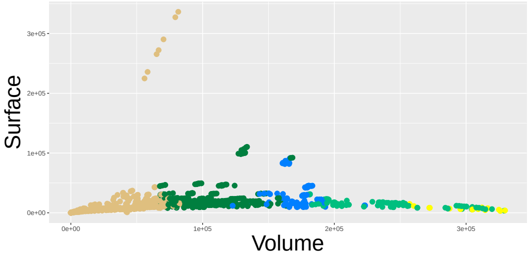
Conjecture

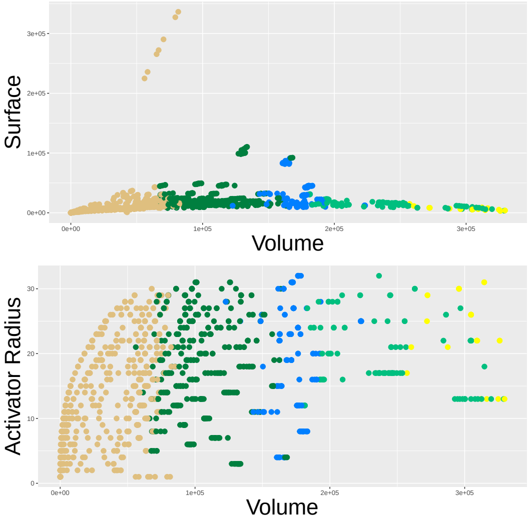
The collections of parameter triplets associated to the structures: "areas", "UC pipes", and "UC spheres" will gain volumetric extent in the parameter space when increasing the maximal values of R_1 and R_2.





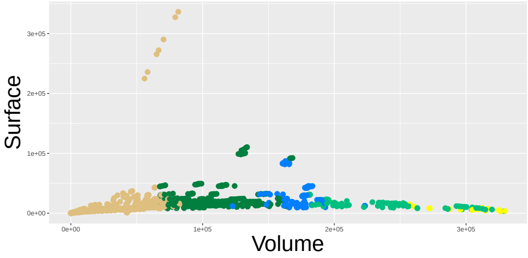
Towards automatic classification

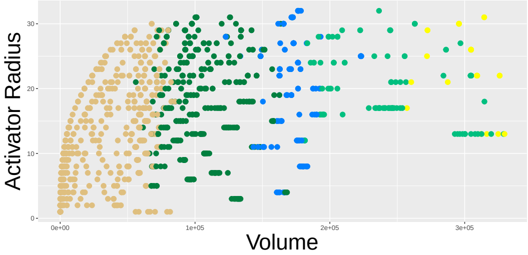


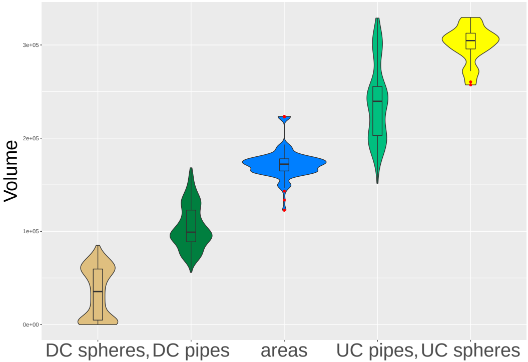


Surface
3e+05
2e+05
1e+05
0e+00
0e+00
1e+05
2e+05
3e+05
Volume
Activator Radius
30
20
10
0
0e+00
1e+05
2e+05
3e+05
Volume

Towards automatic classification



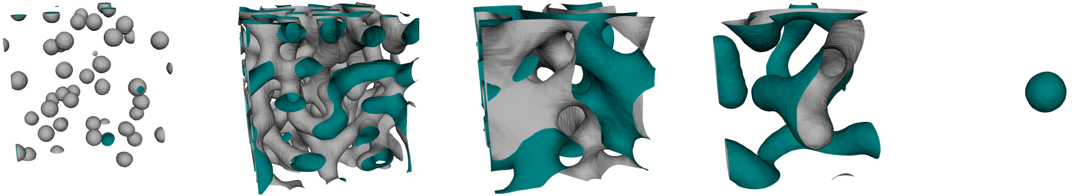




k-NN classification

	Fold 1	**Fold 2**	**Fold 3**	**Fold 4**	**Fold 5**	Error (%)
"DC spheres"	116	115	115	115	115	3.47
"DC pipes"	77	77	77	77	77	8.05
"areas"	29	30	29	30	29	14.97
"UC pipes"	20	20	20	20	21	18.81
"UC spheres"	6	6	5	6	5	42.86
Error (%)	7.26	9.27	8.06	9.68	7.66	

Table: Distribution of the different structures over five folds of the data set used for cross-correlation. The errors given indicate classification errors of the structures solely based on the volume of the structure.

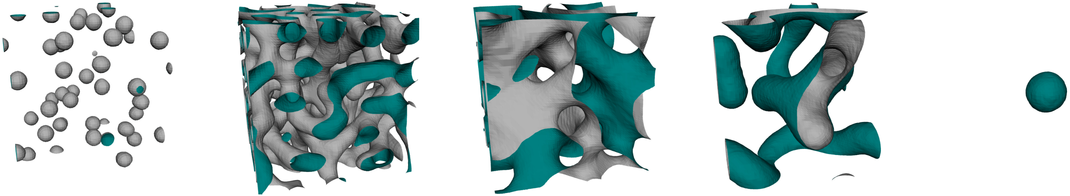


(a) DC spheres "zero-dimensional" (b) DC pipes "one-dimensional" (c) areas "two-dimensional" (d) UC pipes "one-dimensional" (e) UC spheres "zero-dimensional"

Figure: Five of the seven possible Turing-like patterns in a three-torus.

Conjecture

A Turing-like pattern of dimension d exhibits structures of all "dimensions" from 0 to $d-1$. Furthermore, each structure of dimension $0, \ldots, d-2$ occurs twice as "UC" and "DC" version, while structures of "dimension" $d-1$ appear only once.



(a) D

“zero-di

heres

sional”

Thank you for your attention! - Questions?

E-Mail: mail@ms-math-computer.science; **Twitter**: @msmathcomputer2

Conjecture

A Turing-like pattern of dimension d exhibits structures of all “dimensions” from 0 to $d-1$. Furthermore, each structure of dimension $0, \ldots, d-2$ occurs twice as “UC” and “DC” version, while structures of “dimension” $d-1$ appear only once.