Dynamical properties of disjunctive Boolean networks

Maximilien Gadouleau
Durham University

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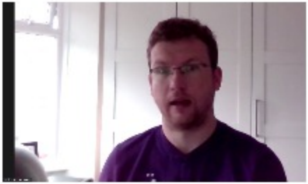

Outline

Disjunction

The Boolean alphabet is $\mathbb{B} = \{0, 1\}$, with the natural order $0 < 1$.
For any $x, y \in \mathbb{B}$, their disjunction is simply

$$x \vee y = \max\{x, y\}.$$

The disjunction

- is associative: $x \vee (y \vee z) = (x \vee y) \vee z$.
- is commutative: $x \vee y = y \vee x$.
- has an identity element, namely 0: $x \vee 0 = 0 \vee x = x$.

So we generalise: let $x = (x_s : s \in S) \in \mathbb{B}^S$, then

$$\bigvee_{s \in S} x_s = \begin{cases} 1 & \text{if } \exists t \in S : x_t = 1 \\ 0 & \text{otherwise.} \end{cases}$$

In particular, if $S = \emptyset$, then $\bigvee_{s \in S} x_s = 0$; if $S = \{s\}$, then $\bigvee_{s \in S} x_s = x_s$.

Boolean network

A point $x \in \mathbb{B}^n$ is called a configuration, which we shall denote as $x = (x_1, \ldots, x_n)$ where $x_i \in \mathbb{B}$ for all $i \in [n] := \{1, \ldots, n\}$.

A Boolean network of dimension n is a mapping

$$f : \mathbb{B}^n \to \mathbb{B}^n.$$

We also split f as $f = (f_1, \ldots, f_n)$ where $f_i : \mathbb{B}^n \to \mathbb{B}$ is a Boolean function representing the update of the state x_i of the i-th entity of the network.

A Boolean network f is disjunctive if f_i is a disjunction for all $i \in [n]$, e.g.

$$f : \mathbb{B}^3 \to \mathbb{B}^3, \qquad f(x) = (x_2, x_1 \vee x_2, 0).$$

Interaction graph

A (directed) graph $D = (V, E)$ is a pair where V is the set of vertices and $E \subseteq V^2$ is the set of arcs of D (Bang-Jensen, Gutin '09).

The interaction graph of f represents the influences of entities on one another: $\mathbb{D}(f) = (V, E)$, where $V = [n]$ and $(i, j) \in E$ iff f_j depends essentially on x_i, i.e. there exists a_{-i} such that

$$f_j(0, a_{-i}) \neq f_j(1, a_{-i}).$$

If D is the interaction graph of f, we then say that f is a Boolean network on D. Clearly, for every graph D there is a unique disjunctive network on D.

Outline

Characterisation of monotone networks

Write $x \le y$ if and only if $x_i \le y_i$ for all $i \in [n]$. We then have

$$x \le y \iff x \vee y = y. \tag{1}$$

A Boolean network f is monotone if $x \le y$ implies $f(x) \le f(y)$; clearly f is monotone if and only if f_i is monotone for all i.

Lemma (Crama, Hammer '11)
A Boolean network f is monotone if and only if for all $x, y \in \mathbb{B}^n$,

$$f(x \vee y) \ge f(x) \vee f(y). \tag{2}$$

We obtain a characterisation of monotone networks based on an equation.

Corollary

A Boolean network f is monotone if and only if for all $x, y \in \mathbb{B}^n$,

$$f(x \vee y) = f(x) \vee f(x \vee y).$$

First characterisation of disjunctive networks

The canonical characterisation of disjunctive networks is as endomorphisms of the disjunction on $\mathbb{B}^n$.

Theorem

A Boolean network f is disjunctive iff $f(0,\ldots,0)=(0,\ldots,0)$ and for all $x,y\in\mathbb{B}^n$,

$$f(x\vee y)=f(x)\vee f(y).$$

Second characterisation

The second characterisation is that disjunctive networks are precisely the submodular monotone networks. A Boolean network is submodular if for all $x, y \in \mathbb{B}^n$,

$$f(x \vee y) \vee f(x \wedge y) \leq f(x) \vee f(y).$$

Submodular Boolean functions form an important class of Horn functions.

Theorem

A Boolean network is disjunctive iff it is monotone and submodular and it fixes the all-zero configuration.

Outline

Representations

Boolean linear mapping

The product of two Boolean matrices is $AB = C$ where

$$c_{ij} = \bigvee_{k=1}^{n} a_{ik} \wedge b_{kj}.$$

A Boolean linear mapping is any $g : \mathbb{B}^n \to \mathbb{B}^n$ of the form $g(x) = xA$ for some Boolean matrix A (Kim ’82).

A graph can be represented by its adjacency matrix A_D where $a_{i,j} = 1$ if and only if $(i,j) \in E$. The disjunctive network on D then satisfies

$$f(x) = xA_D.$$

Since any Boolean matrix is the adjacency matrix of some graph, disjunctive networks are exactly the Boolean linear mappings.

Further representations

Out-neighbourhood function

Identifying $x \in \mathbb{B}^n$ with its support $X := \{i \in [n] : x_i = 1\}$, we can identify f with the mapping on the power set of $[n]$ defined by

$$f(X) = N^{\mathrm{out}}(X).$$

Binary relation

A binary relation R on $[n]$ is a subset of $[n] \times [n]$. The semigroup of binary relations has been widely studied (Howie '95).

Graphs are in one-to-one correspondence with binary relations.

We can then represent f as

$$f(X) = \{y \in [n] : \exists s \in X, (s,y) \in E\}.$$

Outline

Dynamical properties

We consider three kinds of points x for a Boolean network f.

- Image point:
  $$f(y) = x \text{ for some } y \in \mathbb{B}^n.$$
  The number of image points of f is its image rank.
- Periodic point:
  $$f^k(x) = x \text{ for some } k \geq 1.$$
  The number of periodic points of f is its periodic rank.
- Fixed point:
  $$f(x) = x.$$
  The number of fixed points of f is its fixed rank.


Because it fixes 0...0
Knaster-Tarski
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Image points

Given a disjunctive network f and a subset X, it is easy to verify whether X is an image point of f.

Proposition

For any $X \subseteq [n]$, let

$$Y^* := \overline{N^{\mathrm{in}}\left(\overline{X}\right)}.$$

Then X is an image point of f iff $X = N^{\mathrm{out}}(Y^*)$. If so, then

$$Y^* = \bigcup_{Y \in f^{-1}(X)} Y.$$

However, determining the image rank seems difficult.

Problem

What is the complexity of the following problem: given a graph D, what is the image rank of the disjunctive network on D?

Periodic points

We focus on strong graphs D.

The loop number $l(D)$ of D is the greatest common denominator of all the cycle lengths in D.

Lemma (Bang-Jensen, Gutin '09)
If a strong digraph $D = (V, E)$ has loop number $l \geq 2$, then V can be partitioned into $V_0, \ldots, V_{l-1}$ such that $N^{\text{out}}(V_i) = V_{i+1 \mod l}$.

Say a subset of vertices X is D-partite if $X = V_{i_1} \cup \cdots \cup V_{i_a}$ for some $i_1, \ldots, i_a \in [l]$.

Theorem (Goles, Hernández '00)
Let D be a strong graph, then X is a periodic point of the disjunctive network on D iff X is D-partite.

Periodic points

The period of a periodic point X is the smallest $p \geq 1$ such that $f^p(X) = X$.

Corollary (Goles, Hernández '00; Salam Jarrah, Laubenbacher, Veliz-Cuba '10)
The period of a periodic point of f divides $l(D)$. Conversely, for any $p \mid l(D)$, there is a periodic point of period p.

If D is primitive (i.e. D is strong and $l(D) = 1$), then the only periodic points are the two fixed points $X = \emptyset$ and $X = [n]$.

Further results include:

- an upper bound on the time it takes to reach a periodic point (Goles, Hernández '00),
- extension to non-strong graphs (Salam Jarrah, Laubenbacher, Veliz-Cuba '10)
- formula for the number of periodic points of a certain period (Salam Jarrah, Laubenbacher, Veliz-Cuba '10).

Fixed points

Say a graph is nontrivial if every vertex belongs to a cycle.

A collection T of subsets of $[n]$ is a topology on $[n]$ if and only if $\emptyset, [n] \in T$ and

$$X, Y \in T \implies X \cup Y,\ X \cap Y \in T.$$

Let D be a nontrivial graph and for any $S \subseteq [n]$, let the up-set of S to be the set of vertices reachable from S:

$$S^{\uparrow} := \{j : \exists i \in S, \text{there is a path from } i \text{ to } j\}.$$

The up-sets form a topology, denoted

$$\mathbb{T}(D) := \{S^{\uparrow} : S \subseteq [n]\}.$$

Theorem

The set of fixed points of the disjunctive network on the nontrivial graph D is $\mathbb{T}(D)$.

Fixed points

Any finite topology arises from a nontrivial graph; this well known result is usually given in terms of preorders (Cameron '94).

Lemma

Let T be a topology on $[n]$, then $T = \mathbb{T}(D)$ for some nontrivial graph D on $[n]$.

Corollary

T is the set of fixed points of a disjunctive network on a nontrivial graph iff it is a topology.

In comparison, a subset of $\mathbb{B}^n$ is the set of fixed points of a monotone network if and only if it forms a lattice.

Outline

Bijective disjunctive networks

Problem

What is the complexity of the following problem: given k and n, is there a disjunctive network of dimension n with image rank k?

First, we consider the maximum value of the image/periodic/fixed ranks.

A permutation of variables is any of the form

$$\pi(x) = (x_{\pi(1)}, \dots, x_{\pi(n)})$$

for some permutation π of $[n]$.

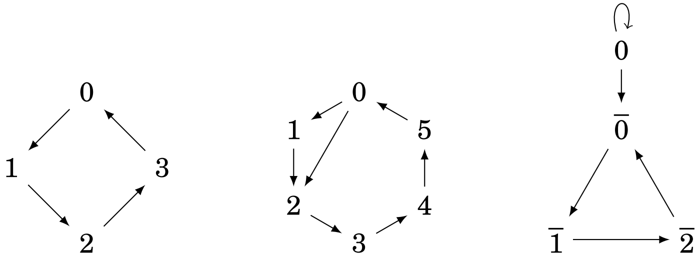
Theorem (Folklore)

A monotone Boolean network is bijective iff it is a permutation of variables.

Near-bijective networks

We now move on to the largest rank of singular networks. We will need these graphs.

1. C_n ($n \geq 1$) is the cycle on n vertices.
2. $A_{p,q}$ ($0 \leq q \leq p-1$) is C_p, augmented by the chord $\{(0,q)\}$.
3. $B_{s,t}$ ($s,t \geq 1$) is the link of cycles C_s and C_t, with a single arc from C_s to C_t.



(a) C_4 (b) $A_{6,2}$ (c) $B_{1,3}$

Near-bijective networks

Say a graph is near-cyclic if one of its connected components is a chorded cycle or a link of cycles, and all other connected components are cycles.

Theorem

For $n \geq 2$, the maximum image, periodic, and fixed rank of a singular disjunctive network of dimension n is $3/4 \cdot 2^n$, and it is reached iff its interaction graph:

- *(image rank) is near-cyclic.*
- *(periodic rank) is of the form $B_{1,1} \cup C_{n_1} \cup \cdots \cup C_{n_c}$.*
- *(fixed rank) is of the form $B_{1,1} \cup C_1 \cup \cdots \cup C_1$.*

Lowest ranks

We next consider the opposite problem: what is the smallest "missing value" of the image/periodic/fixed rank?

Problem

Given n, what is the minimum $k \geq 1$ such that there is no disjunctive network of dimension n and image / periodic / fixed rank k?

We give a lower bound on that quantity below, for all three ranks.

Theorem

For any n, there exists an idempotent disjunctive network on n vertices with image, periodic, and fixed ranks r for all $1 \leq r \leq p-1$, where p is the smallest prime number greater than $n+1$.

Comparing the disjunctive network to other networks

Can the disjunctive network maximise the rank?

Theorem (G '18; Riis '06; Aracena '08)

Let f be a Boolean network with interaction graph D.

1. the image rank of f is at most $2^{\alpha_1(D)}$.
2. the periodic rank of f is at most $2^{\alpha_n(D)}$.
3. the fixed rank of f is at most $2^{\tau(D)}$.

(The third item is the famous feedback bound!)

Problem

Classify the graphs D such that:

1. *the image rank of the disjunctive network on D is $2^{\alpha_1(D)}$.*
2. *the periodic rank of the disjunctive network on D is $2^{\alpha_n(D)}$.*
3. *the fixed rank of the disjunctive network on D is $2^{\tau(D)}$.*

Comparing the disjunctive network to other networks

When does the disjunctive network minimise the image / periodic / fixed rank?

Problem

Classify the graphs D such that the disjunctive network minimises the image/periodic rank over all Boolean networks on D.

For the image rank, (G '20) gives an example, where the minimum image rank is not achieved by the disjunctive network.

For the periodic rank, we do not even know which graphs D admit a Boolean network with a single periodic point (the so-called nilpotent networks) (G, Richard '16).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Thank you!

Merci !