Algebraic structures hiding in automata networks

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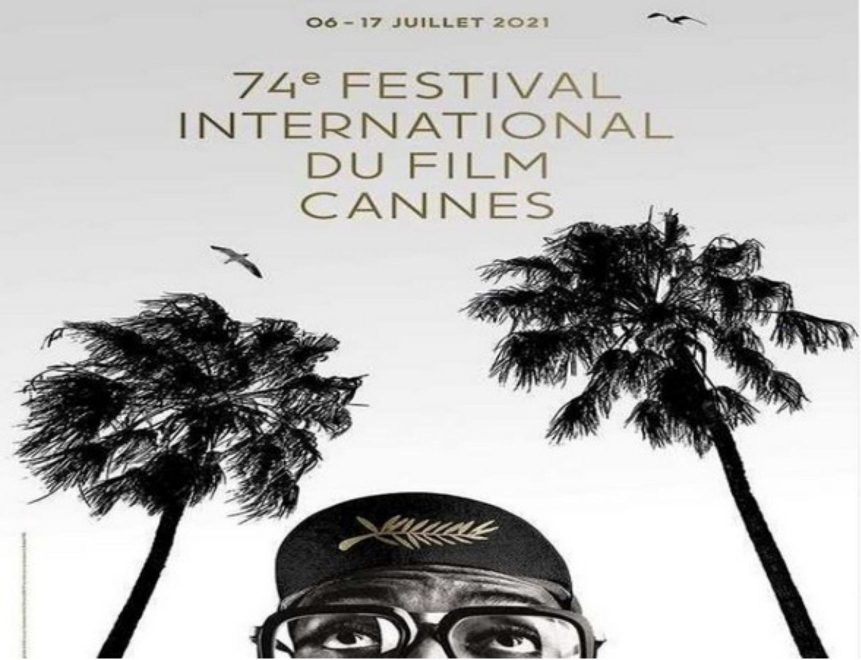
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The problem: Toggling independent sets

An independent set of a graph is a subset of vertices such that no two are adjacent.

Let $\mathcal{I}_n$ denote the independent sets, written as binary strings.

At each vertex k the toggle operation T_k is the bijection

$$T_k \colon \mathcal{I}_n \longrightarrow \mathcal{I}_n, \qquad T_k(E) = \begin{cases} E \cup \{k\} & k \notin E \text{ and } E \cup \{k\} \in \mathcal{I}_n \\ E \setminus \{k\} & k \in E \text{ and } E \setminus \{k\} \in \mathcal{I}_n \\ E & \text{otherwise.} \end{cases}$$

We are interested in understanding the dynamics of the map

$$F_\pi := T_{\pi(n)} \circ \cdots \circ T_{\pi(2)} \circ T_{\pi(1)},$$

for some fixed permutation $\pi \in \mathcal{S}_n$.

This is an asynchronous automata network, or sequential dynamical system.

In this talk, we'll only consider the cycle graph Circ_n, which defines an asynchronous cellular automaton, for "ECA rule 1".

Additionally, we'll update in a "fixed sweep": $\pi = 123 \cdots n$.

The original toggling problem: order ideals of posets

"down sets"

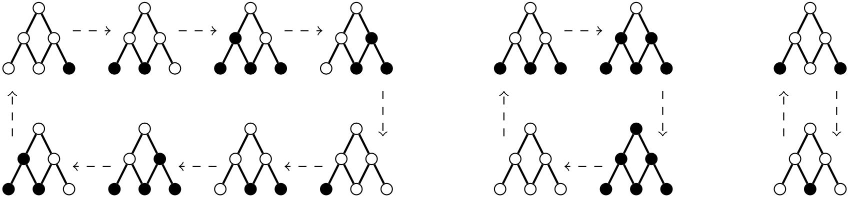


Figure: Rowmotion – toggle "*by rows, top-to-bottom*".

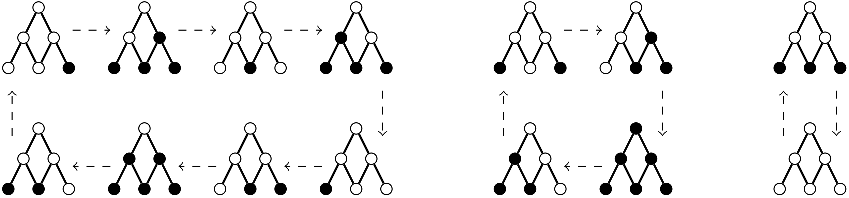


Figure: Promotion – toggle "*by columns, left-to-right*".

Background and (approximate) timeline of this project

- 2008: Order independence, dynamics groups (M– w/ J. McCammond & H.S. Mortveit)
- 2012: First toggle paper, order ideals (Jessica Striker & Nathan Williams)
- 2014: SDS project @ UC Santa Barbara (Colin Defant)
- 2015: Toggling non-crossing partitions (with Mike Joseph, et al.)
- 2016: Block invariance for ECA (Eric Goles, Marco Montalva, Henning Mortveit, et al.)
- 2018: Antichain toggling (Mike Joseph)
- 2019: Toggling independent sets over path graphs (Mike Joseph)
- 2020: DAC meeting @ Banff Research Station, Canada (birth of our project!)
- 2021: Toggle group, independent sets over cycle graphs (Y. Numata & Y. Yamanouchi)

Figure: Colin Defant, Mike Joseph, and Alex McDonough

Let's see some examples...

$x_{i-1}x_ix_{i+1}$	111	110	101	100	011	010	001	000
$T_i(x_{i-1}, x_i, x_{i+1})$	N/A	N/A	0	0	N/A	0	0	1

x	v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8	v_9	v_{10}	v_{11}	v_{12}
	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
$x^{(0)}$	0	0	0	0	0	0	**1**	0	**1**	0	0	**1**
$x^{(1)}$	0	**1**	0	**1**	0	0	0	0	0	**1**	0	0
$x^{(2)}$	0	0	0	0	**1**	0	**1**	0	0	0	**1**	0
$x^{(3)}$	**1**	0	**1**	0	0	0	0	**1**	0	0	0	0
$x^{(4)}$	0	0	0	**1**	0	**1**	0	0	**1**	0	**1**	0
$x^{(5)}$	**1**	0	0	0	0	0	**1**	0	0	0	0	0
$x^{(6)}$	0	**1**	0	**1**	0	0	0	**1**	0	**1**	0	**1**
$x^{(7)}$	0	0	0	0	**1**	0	0	0	0	0	0	0
$x^{(8)}$	**1**	0	**1**	0	0	**1**	0	**1**	0	**1**	0	0
$x^{(9)}$	0	0	0	**1**	0	0	0	0	0	0	**1**	0
$x^{(10)}$	**1**	0	0	0	**1**	0	**1**	0	**1**	0	0	0
$x^{(11)}$	0	**1**	0	0	0	0	0	0	0	**1**	0	**1**
$x^{(12)}$	0	0	**1**	0	**1**	0	**1**	0	0	0	0	0
$x^{(13)}$	**1**	0	0	0	0	0	0	**1**	0	**1**	0	0
$x^{(14)}$	0	**1**	0	**1**	0	**1**	0	0	0	0	**1**	0
	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
sum	5	4	3	5	4	3	5	4	3	5	4	3

$n = 12$ nodes, orbit size $m = 15$, $nm = 180$ entries, period $T = 45$, frequency $\omega = 4$.

Let's see some examples. . .

2 snakes

6 co-snakes

x	v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8	v_9	v_{10}	v_{11}
	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
$x^{(0)}$	0	0	0	0	1	0	1	0	0	0	0
$x^{(1)}$	1	0	1	0	0	0	0	1	0	1	0
$x^{(2)}$	0	0	0	1	0	1	0	0	0	0	1
$x^{(3)}$	0	1	0	0	0	0	1	0	1	0	0
$x^{(4)}$	0	0	1	0	1	0	0	0	0	1	0
$x^{(5)}$	1	0	0	0	0	1	0	1	0	0	0
$x^{(6)}$	0	1	0	1	0	0	0	0	1	0	1
$x^{(7)}$	0	0	0	0	1	0	1	0	0	0	0
$x^{(8)}$	1	0	1	0	0	0	0	1	0	1	0
$x^{(9)}$	0	0	0	1	0	1	0	0	0	0	1
$x^{(10)}$	0	1	0	0	0	0	1	0	1	0	0
$x^{(11)}$	0	0	1	0	1	0	0	0	0	1	0
$x^{(12)}$	1	0	0	0	0	1	0	1	0	0	0
$x^{(13)}$	0	1	0	1	0	0	0	0	1	0	1
$x^{(14)}$	0	0	0	0	1	0	1	0	0	0	0
$x^{(15)}$	1	0	1	0	0	0	0	1	0	1	0
$x^{(16)}$	0	0	0	1	0	1	0	0	0	0	1
$x^{(17)}$	0	1	0	0	0	0	1	0	1	0	0
$x^{(18)}$	0	0	1	0	1	0	0	0	0	1	0
$x^{(19)}$	1	0	0	0	0	1	0	1	0	0	0
$x^{(20)}$	0	1	0	1	0	0	0	0	1	0	1
	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮

"slither"

2D2D2D

$= (2D)^3$

$n = 11$ nodes, orbit size $m = 7$, $nm = 77$ entries, period $T = 7$, frequency $\omega = 11$.

Some curious observations

- The "sum vector" seems to always have odd period (but not for other update orders).
- The diagonal patterns of 1s seems to have the same structure in a given orbit.
- The orbits tend to be made up of several repeating copies of a certain binary string.
- For each n, some orbit sizes come up more often than others.

Idea

All of these features, and more, can be understood algebraically.

Reminder!

Everything we're doing in this talk is only for ECA rule 1.

Exercise for the audience

Think about how these ideas might carry over (under modifications) to other ECA rules.

Scrolls and ticker tapes

We'll use two ways to view the dynamics, starting from $x = x^{(0)} \in \mathbb{F}_2^n$.

- The scroll, a $\mathbb{Z} \times \mathbb{Z}_n$ binary grid, denoted $\mathcal{S} = \mathsf{Scroll}(x) = (X_{i,j})$
- The ticker tape, a bi-infinite binary sequence, denoted $\mathcal{X} = \mathsf{Tape}(x) = (X_k)$.

In both notations, the live entries are those that are 1:

$$\mathsf{Live}(\mathcal{S}) = \big\{(i,j) \in \mathbb{Z} \times \mathbb{Z}_n \mid X_{i,j} = 1\big\}, \qquad \mathsf{Live}(\mathcal{X}) = \big\{k \in \mathbb{Z} \mid X_k = 1\big\}.$$

The shadow and successor functions

For any live entry (i,j) in $\mathcal{S}$, there are two "complementary pairs" of live entries:

$$\begin{array}{ccccccc} \cdots & 0 & 0 & \cdots & & & \\ \cdots & 0 & 1 & 0 & a & \cdots & \\ \cdots & 0 & 0 & \bar{a} & 0 & \cdots & \end{array}$$

"2 vs. D"

"the successor, $s(i,j)$"

$$\begin{array}{cccccc} \cdots & & 0 & 0 & \cdots & \\ \cdots & 0 & 1 & 0 & \cdots & \\ \cdots & 0 & 0 & \cdots & & \\ \cdots & b & \bar{b} & \cdots & & \end{array}$$

"L vs. S"

"the shadow, $c(i,j)$"

Definition

Given a live entry $(i,j) \in \mathcal{S}$, the snake and the co-snake containing it are:

$$\mathrm{Snake}(i,j) = \{s^k(i,j) \mid k \in \mathbb{Z}\}, \qquad \mathrm{CoSnake}(i,j) = \{c^k(i,j) \mid k \in \mathbb{Z}\}.$$

x	v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8	v_9	v_{10}	v_{11}
	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
$x^{(0)}$	0	0	0	0	1	0	1	0	0	0	0
$x^{(1)}$	1	0	1	0	0	0	0	1	0	1	0
$x^{(2)}$	0	0	0	1	0	1	0	0	0	0	1
$x^{(3)}$	0	1	0	0	0	0	1	0	1	0	0
$x^{(4)}$	0	0	1	0	1	0	0	0	0	1	0
$x^{(5)}$	1	0	0	0	0	1	0	1	0	0	0
$x^{(6)}$	0	1	0	1	0	0	0	0	1	0	1
	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮

x	v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8	v_9	v_{10}	v_{11}
	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
$x^{(0)}$	0	0	0	0	1	0	1	0	0	0	0
$x^{(1)}$	1	0	1	0	0	0	0	1	0	1	0
$x^{(2)}$	0	0	0	1	0	1	0	0	0	0	1
$x^{(3)}$	0	1	0	0	0	0	1	0	1	0	0
$x^{(4)}$	0	0	1	0	1	0	0	0	0	1	0
$x^{(5)}$	1	0	0	0	0	1	0	1	0	0	0
$x^{(6)}$	0	1	0	1	0	0	0	0	1	0	1
	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮

The snake group

The successor and shadow functions are bijections on Live($\mathcal{S}$). They define the infinite "snake group"

$$G(\mathcal{S}) := \langle s, c \rangle.$$

Proposition

The successor and shadow functions commute.

Proof (sketch)

Check all cases:

$$\begin{array}{c} \cdots\ 0\ 0\ \cdots \\ \cdots\ 0\ \mathbf{1}\ 0\ \mathbf{a}\ \cdots \\ \cdots\ 0\ 0\ 0\ \overline{\mathbf{a}}\ 0\ \cdots \\ \cdots\ 0\ \mathbf{1}\ 0\ \mathbf{b}\ \cdots \\ \cdots\ 0\ 0\ \overline{\mathbf{b}}\ 0\ \cdots \end{array} \qquad \begin{array}{c} \cdots\ 0\ 0\ \cdots \\ \cdots\ 0\ \mathbf{1}\ 0\ \mathbf{a}\ \cdots \\ \cdots\ 0\ 0\ \overline{\mathbf{a}}\ 0\ \cdots \\ \cdots\ 0\ \mathbf{1}\ 0\ \mathbf{b}\ \cdots \\ \cdots\ 0\ 0\ \overline{\mathbf{b}}\ 0\ \cdots \end{array}$$

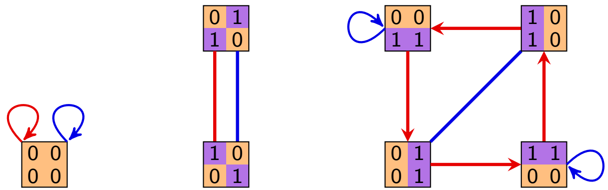
Our first goal is to find a presentation:

$$G(\mathcal{S}) := \langle s, c \mid sc = cs,\ ???\rangle.$$

We'll do this by considering the action of $G(\mathcal{S})$ on Live($\mathcal{S}$).

Group action crash course

Example of the dihedral group $D_4 = \langle r, f \mid r^4 = f^2 = 1,\ rfr = f \rangle$ acting on a size-7 set:



Types of actions:

- Transitive: "one connected component (i.e., orbit)"
- Free: "every loop is the identity element"
- Simply transitive: "transitive & free" [bijective correspondence $G \leftrightarrow X$!]

Our approach

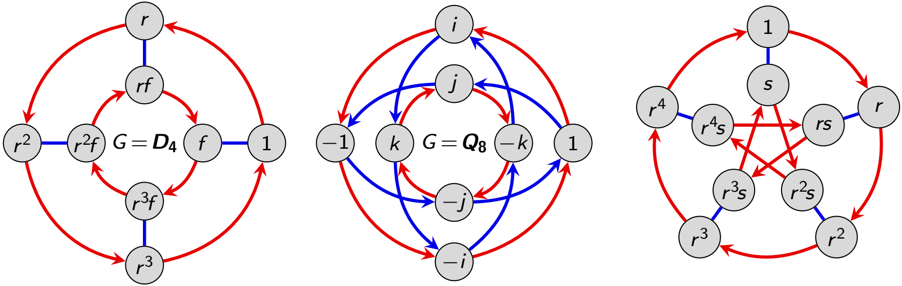
Study the action of the snake group $G(\mathcal{S}) = \langle s, c \rangle$ on the set $\text{Live}(\mathcal{S})$ of live entries.

Simply transitive actions and Cayley diagrams

Simply transitive actions are very nice!

They endow the set with the structure of a Cayley diagram – a "map of a group."

Example. Two Cayley diagrams and an imposter:

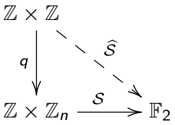


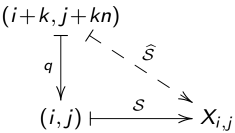
If G acts simply transitively on X, then we say that X is a G-torsor.

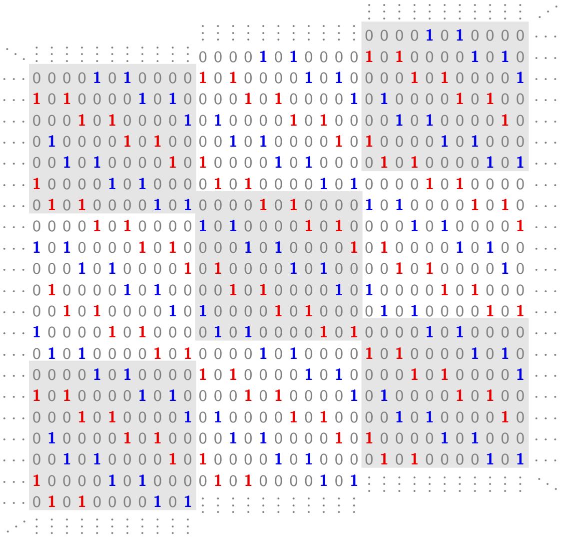
Examples include Lie groups, Coxeter groups, sandpile groups, fundamental groups, etc.

Lifting the scroll $\mathcal{S}\colon \mathbb{Z}\times\mathbb{Z}_n \to \mathbb{F}_2$, to the universal scroll $\widehat{\mathcal{S}}\colon \mathbb{Z}\times\mathbb{Z}\to\mathbb{F}_2$

It can be helpful work work "upstairs":







Snakes and co-snakes on $\mathbb{R}^2$.

The shadow and successor functions lift to the universal cover:

$$\begin{array}{ccc} \mathrm{Live}(\widehat{\mathcal{S}}) & \xrightarrow{\ \widehat{s}\ } & \mathrm{Live}(\widehat{\mathcal{S}}) \\ \downarrow q & & \downarrow q \\ \mathrm{Live}(\mathcal{S}) & \xrightarrow{\ s\ } & \mathrm{Live}(\mathcal{S}) \end{array} \qquad \begin{array}{ccc} (i+k, j+kn) & \stackrel{\widehat{s}}{\longmapsto} & \widehat{s}(i+k, j+kn) \\ \downarrow q & & \downarrow q \\ (i,j) & \stackrel{s}{\longmapsto} & s(i,j) \end{array}$$

Definition

Given a live entry $(i,j) \in \widehat{\mathcal{S}}$, the affine snake and affine co-snake containing it are:

$$\mathrm{Snake}^{\text{✈}}(i,j) = \{\widehat{s}^k(i,j) \mid k \in \mathbb{Z}\}, \qquad \mathrm{CoSnake}^{\text{✈}}(i,j) = \{\widehat{c}^k(i,j) \mid k \in \mathbb{Z}\}.$$

Affine snake lemma

The affine snake group $G(\widehat{\mathcal{S}}) := \langle \widehat{s}, \widehat{c} \rangle$ is free abelian, and acts simply transitively on $\mathrm{Live}(\widehat{\mathcal{S}})$.

Corollary

Affine snakes are cosets of $\langle \widehat{s} \rangle$, and affine co-snakes are cosets of $\langle \widehat{c} \rangle$.

Slithers and co-slithers

Suppose that $\mathcal{S}$ has α snakes and β co-snakes.

Proposition

The snake group $G(\mathcal{S}) := \langle s, c \rangle$ acts simply transitively on $\text{Live}(\mathcal{S})$, and has presentation

$$G(\mathcal{S}) = \langle s, c \mid sc = cs,\ s^\beta = c^\alpha \rangle.$$

Corollary

Snakes are cosets of $\langle s \rangle$, and co-snakes are cosets of $\langle c \rangle$.

Denote the "shape" of a snake by a length-β sequence of Ds and 2s called the slither.

Denote the "shape" of a co-snake by a length-α sequence of Ss and Ls called the co-slither.

Co-snake lemma

In any scroll, all (co-)snakes have the same (co-)slither.

Some more examples

"scale" = 42

0	0	0	0	**1**	0	**1**	0	0	0	0
1	0	**1**	0	0	0	0	**1**	0	**1**	0
0	0	0	**1**	0	**1**	0	0	0	0	**1**
0	**1**	0	0	0	0	**1**	0	**1**	0	0
0	0	**1**	0	**1**	0	0	0	0	**1**	0
1	0	0	0	0	**1**	0	**1**	0	0	0
0	**1**	0	**1**	0	0	0	0	**1**	0	**1**

$n = 11$, $m = 7$, $nm = 77$
period $T = 7$, frequency $\omega = 11$
$\alpha = 2$ snakes, $\beta = 6$ co-snakes
slither: $(2D)^3$, co-slither: S^2

0	0	0	**1**	0	0	0	**1**	0	0	0
1	0	0	0	**1**	0	0	0	**1**	0	0
0	**1**	0	0	0	**1**	0	0	0	**1**	0
0	0	**1**	0	0	0	**1**	0	0	0	**1**

$n = 11$, $m = 4$, $nm = 44$
period $T = 4$, frequency $\omega = 11$
$\alpha = 3$ snakes, $\beta = 5$ co-snakes
slither: D^5, co-slither L^3

0	0	**1**	0	0	**1**	0	0	**1**	0	0
1	0	0	**1**	0	0	**1**	0	0	**1**	0
0	**1**	0	0	**1**	0	0	**1**	0	0	**1**

$n = 11$, $m = 3$, $nm = 33$
period $T = 3$, frequency $\omega = 11$
$\alpha = 4$ snakes, $\beta = 7$ co-snakes
slither: D^7, co-slither S^3

Quotienting scrolls to tables

$$\begin{array}{ccc} \mathrm{Live}(\mathcal{S}) & \xrightarrow{\;s\;} & \mathrm{Live}(\mathcal{S}) \\ \big\downarrow{\scriptstyle p} & & \big\downarrow{\scriptstyle p} \\ \mathrm{Live}(\mathcal{T}) & \xrightarrow{\;\bar{s}\;} & \mathrm{Live}(\mathcal{T}) \end{array} \qquad\qquad \begin{array}{ccc} (i+km,j) & \overset{s}{\longmapsto} & s(i+km,j) \\ \big\downarrow{\scriptstyle p} & & \big\downarrow{\scriptstyle p} \\ (i,j) & \overset{\bar{s}}{\longmapsto} & \bar{s}(i,j) \end{array}$$

This causes snakes to wrap from bottom-to-top, like an ouroboros:

Example: a fundamental and 2-fold orbit table.

x	v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8	v_9	v_{10}	v_{11}
$x^{(0)}$	0	0	0	0	1	0	1	0	0	0	0
$x^{(1)}$	1	0	1	0	0	0	0	1	0	1	0
$x^{(2)}$	0	0	0	1	0	1	0	0	0	0	1
$x^{(3)}$	0	1	0	0	0	0	1	0	1	0	0
$x^{(4)}$	0	0	1	0	1	0	0	0	0	1	0
$x^{(5)}$	1	0	0	0	0	1	0	1	0	0	0
$x^{(6)}$	0	1	0	1	0	0	0	0	1	0	1

x	v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8	v_9	v_{10}	v_{11}
$x^{(0)}$	0	0	0	0	1	0	1	0	0	0	0
$x^{(1)}$	1	0	1	0	0	0	0	1	0	1	0
$x^{(2)}$	0	0	0	1	0	1	0	0	0	0	1
$x^{(3)}$	0	1	0	0	0	0	1	0	1	0	0
$x^{(4)}$	0	0	1	0	1	0	0	0	0	1	0
$x^{(5)}$	1	0	0	0	0	1	0	1	0	0	0
$x^{(6)}$	0	1	0	1	0	0	0	0	1	0	1

x	v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8	v_9	v_{10}	v_{11}
$x^{(0)}$	0	0	0	0	1	0	1	0	0	0	0
$x^{(1)}$	1	0	1	0	0	0	0	1	0	1	0
$x^{(2)}$	0	0	0	1	0	1	0	0	0	0	1
$x^{(3)}$	0	1	0	0	0	0	1	0	1	0	0
$x^{(4)}$	0	0	1	0	1	0	0	0	0	1	0
$x^{(5)}$	1	0	0	0	0	1	0	1	0	0	0
$x^{(6)}$	0	1	0	1	0	0	0	0	1	0	1
$x^{(7)}$	0	0	0	0	1	0	1	0	0	0	0
$x^{(8)}$	1	0	1	0	0	0	0	1	0	1	0
$x^{(9)}$	0	0	0	1	0	1	0	0	0	0	1
$x^{(10)}$	0	1	0	0	0	0	1	0	1	0	0
$x^{(11)}$	0	0	1	0	1	0	0	0	0	1	0
$x^{(12)}$	1	0	0	0	0	1	0	1	0	0	0
$x^{(13)}$	0	1	0	1	0	0	0	0	1	0	1

x	v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8	v_9	v_{10}	v_{11}
$x^{(0)}$	0	0	0	0	1	0	1	0	0	0	0
$x^{(1)}$	1	0	1	0	0	0	0	1	0	1	0
$x^{(2)}$	0	0	0	1	0	1	0	0	0	0	1
$x^{(3)}$	0	1	0	0	0	0	1	0	1	0	0
$x^{(4)}$	0	0	1	0	1	0	0	0	0	1	0
$x^{(5)}$	1	0	0	0	0	1	0	1	0	0	0
$x^{(6)}$	0	1	0	1	0	0	0	0	1	0	1
$x^{(7)}$	0	0	0	0	1	0	1	0	0	0	0
$x^{(8)}$	1	0	1	0	0	0	0	1	0	1	0
$x^{(9)}$	0	0	0	1	0	1	0	0	0	0	1
$x^{(10)}$	0	1	0	0	0	0	1	0	1	0	0
$x^{(11)}$	0	0	1	0	1	0	0	0	0	1	0
$x^{(12)}$	1	0	0	0	0	1	0	1	0	0	0
$x^{(13)}$	0	1	0	1	0	0	0	0	1	0	1

Example: a 3-fold orbit table.

x	v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8	v_9	v_{10}	v_{11}
$x^{(0)}$	0	0	0	0	**1**	0	**1**	0	0	0	0
$x^{(1)}$	**1**	0	**1**	0	0	0	0	**1**	0	**1**	0
$x^{(2)}$	0	0	0	**1**	0	**1**	0	0	0	0	**1**
$x^{(3)}$	0	**1**	0	0	0	0	**1**	0	**1**	0	0
$x^{(4)}$	0	0	**1**	0	**1**	0	0	0	0	**1**	0
$x^{(5)}$	**1**	0	0	0	0	**1**	0	**1**	0	0	0
$x^{(6)}$	0	**1**	0	**1**	0	0	0	0	**1**	0	**1**
$x^{(7)}$	0	0	0	0	**1**	0	**1**	0	0	0	0
$x^{(8)}$	**1**	0	**1**	0	0	0	0	**1**	0	**1**	0
$x^{(9)}$	0	0	0	**1**	0	**1**	0	0	0	0	**1**
$x^{(10)}$	0	**1**	0	0	0	0	**1**	0	**1**	0	0
$x^{(11)}$	0	0	**1**	0	**1**	0	0	0	0	**1**	0
$x^{(12)}$	**1**	0	0	0	0	**1**	0	**1**	0	0	0
$x^{(13)}$	0	**1**	0	**1**	0	0	0	0	**1**	0	**1**
$x^{(14)}$	0	0	0	0	**1**	0	**1**	0	0	0	0
$x^{(15)}$	**1**	0	**1**	0	0	0	0	**1**	0	**1**	0
$x^{(16)}$	0	0	0	**1**	0	**1**	0	0	0	0	**1**
$x^{(17)}$	0	**1**	0	0	0	0	**1**	0	**1**	0	0
$x^{(18)}$	0	0	**1**	0	**1**	0	0	0	0	**1**	0
$x^{(19)}$	**1**	0	0	0	0	**1**	0	**1**	0	0	0
$x^{(20)}$	0	**1**	0	**1**	0	0	0	0	**1**	0	**1**

x	v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8	v_9	v_{10}	v_{11}
$x^{(0)}$	0	0	0	0	**1**	0	**1**	0	0	0	0
$x^{(1)}$	**1**	0	**1**	0	0	0	0	**1**	0	**1**	0
$x^{(2)}$	0	0	0	**1**	0	**1**	0	0	0	0	**1**
$x^{(3)}$	0	**1**	0	0	0	0	**1**	0	**1**	0	0
$x^{(4)}$	0	0	**1**	0	**1**	0	0	0	0	**1**	0
$x^{(5)}$	**1**	0	0	0	0	**1**	0	**1**	0	0	0
$x^{(6)}$	0	**1**	0	**1**	0	0	0	0	**1**	0	**1**
$x^{(7)}$	0	0	0	0	**1**	0	**1**	0	0	0	0
$x^{(8)}$	**1**	0	**1**	0	0	0	0	**1**	0	**1**	0
$x^{(9)}$	0	0	0	**1**	0	**1**	0	0	0	0	**1**
$x^{(10)}$	0	**1**	0	0	0	0	**1**	0	**1**	0	0
$x^{(11)}$	0	0	**1**	0	**1**	0	0	0	0	**1**	0
$x^{(12)}$	**1**	0	0	0	0	**1**	0	**1**	0	0	0
$x^{(13)}$	0	**1**	0	**1**	0	0	0	0	**1**	0	**1**
$x^{(14)}$	0	0	0	0	**1**	0	**1**	0	0	0	0
$x^{(15)}$	**1**	0	**1**	0	0	0	0	**1**	0	**1**	0
$x^{(16)}$	0	0	0	**1**	0	**1**	0	0	0	0	**1**
$x^{(17)}$	0	**1**	0	0	0	0	**1**	0	**1**	0	0
$x^{(18)}$	0	0	**1**	0	**1**	0	0	0	0	**1**	0
$x^{(19)}$	**1**	0	0	0	0	**1**	0	**1**	0	0	0
$x^{(20)}$	0	**1**	0	**1**	0	0	0	0	**1**	0	**1**

The ouroboros group

The successor and shadow functions descend to bijections

$$\bar{s}, \bar{c} \colon \mathrm{Live}(\mathcal{T}) \longrightarrow \mathrm{Live}(\mathcal{T}).$$

$$\begin{array}{ccc} \mathrm{Live}(\mathcal{S}) & \xrightarrow{s} & \mathrm{Live}(\mathcal{S}) \\ \downarrow p & & \downarrow p \\ \mathrm{Live}(\mathcal{T}) & \xrightarrow{\bar{s}} & \mathrm{Live}(\mathcal{T}) \end{array} \qquad \begin{array}{ccc} (i+km, j) & \overset{s}{\longmapsto} & s(i+km, j) \\ \downarrow p & & \downarrow p \\ (i,j) & \overset{\bar{s}}{\longmapsto} & \bar{s}(i,j) \end{array}$$

Definition

Given a live entry $(i,j) \in \mathcal{T}$, the ouroboros and co-ouroboros containing it are

$$\mathrm{Ouro}(i,j) = \{\bar{s}^k(i,j) \mid k \in \mathbb{Z}\}, \qquad \mathrm{CoOuro}(i,j) = \{\bar{c}^k(i,j) \mid k \in \mathbb{Z}\}.$$

Suppose $\mathcal{S}$ has α snakes, β co-snakes and its ω-fold orbit table $\mathcal{T}$ has $\bar{\alpha}$ ouroboroi, $\bar{\beta}$ co-ouroboroi, and τ live entries. The ouroboros group $G(\mathcal{T}) := \langle \bar{s}, \bar{c} \rangle$ has presentation

$$G(\mathcal{T}) = \left\langle \bar{s}, \bar{c} \;\middle|\; \bar{s}\,\bar{c} = \bar{c}\,\bar{s},\ \bar{s}^{\beta} = \bar{c}^{\alpha},\ \bar{s}^{\,\tau/\bar{\alpha}} = \bar{c}^{\,\tau/\bar{\beta}} = 1 \right\rangle \cong \mathbb{Z}_{\bar{\alpha}} \times \mathbb{Z}_{\tau/\bar{\alpha}} \cong \mathbb{Z}_{\bar{\beta}} \times \mathbb{Z}_{\tau/\bar{\beta}},$$

and it acts simply transitively on $\mathrm{Live}(\mathcal{T})$.

My favorite result

The map $p\colon \mathrm{Live}(\mathcal{S}) \longrightarrow \mathrm{Live}(\mathcal{T})$ is a topological covering map.

It induces a homomorphism

$$p^*\colon G(\mathcal{S}) \longrightarrow G(\mathcal{T}).$$

Definition

The (co-)ouroboros degree of is the number of (co-)snakes in the p-preimage of each (co-)ouroboros:

$$\deg(p^*) := \frac{[G(\mathcal{S}) : \langle s\rangle]}{[G(\mathcal{T}) : \langle \bar{s}\rangle]} = \alpha/\bar{\alpha}, \qquad \operatorname{codeg}(p^*) := \frac{[G(\mathcal{S}) : \langle c\rangle]}{[G(\mathcal{T}) : \langle \bar{c}\rangle]} = \beta/\bar{\beta}.$$

Theorem

The fundamental period of a scroll is

$$T(\mathcal{S}) = \frac{T(\mathcal{X})}{\gcd(T(\mathcal{X}), n)} = \frac{\mathrm{Scale}(\mathcal{X})}{\deg(p^*)\operatorname{codeg}(p^*)\gcd(T(\mathcal{X}), n)},$$

where $\mathrm{Scale}(\mathcal{X}) = s^{\beta}(k) - k = c^{\alpha}(k) - k$, for any $k \in \mathbb{Z}$.

Some familiar examples for $n = 11$

0	0	0	0	**1**	0	**1**	0	0	0	0
1	0	**1**	0	0	0	0	**1**	0	**1**	0
0	0	0	**1**	0	**1**	0	0	0	0	**1**
0	**1**	0	0	0	0	**1**	0	**1**	0	0
0	0	**1**	0	**1**	0	0	0	0	**1**	0
1	0	0	0	0	**1**	0	**1**	0	0	0
0	**1**	0	**1**	0	0	0	0	**1**	0	**1**

$\alpha = 2$ snakes, $\beta = 6$ co-snakes

$\bar{\alpha} = 2$ ouroboroi, $\bar{\beta} = 2$ co-ouroboroi

$\deg(p_*) = 1$, $\operatorname{codeg}(p_*) = 3$

$T(\mathcal{X}) = 7$, $\operatorname{Scale}(\mathcal{X}) = 42$

$$T(\mathcal{S}) = 7 = \frac{42}{2 \cdot 3 \cdot 1}$$

0	0	0	**1**	0	0	0	**1**	0	0	0
1	0	0	0	**1**	0	0	0	**1**	0	0
0	**1**	0	0	0	**1**	0	0	0	**1**	0
0	0	**1**	0	0	0	**1**	0	0	0	**1**
0	0	0	**1**	0	0	0	**1**	0	0	0
1	0	0	0	**1**	0	0	0	**1**	0	0
0	**1**	0	0	0	**1**	0	0	0	**1**	0
0	0	**1**	0	0	0	**1**	0	0	0	**1**

$\alpha = 3$ snakes, $\beta = 5$ co-snakes

$\bar{\alpha} = 1$ ouroboros, $\bar{\beta} = 1$ co-ouroboros

$\deg(p_*) = 3$, $\operatorname{codeg}(p_*) = 5$

$T(\mathcal{X}) = 4$, $\operatorname{Scale}(\mathcal{X}) = 60$

$$T(\mathcal{S}) = 4 = \frac{60}{3 \cdot 5 \cdot 1}$$

0	0	**1**	0	0	**1**	0	0	**1**	0	0
1	0	0	**1**	0	0	**1**	0	0	**1**	0
0	**1**	0	0	**1**	0	0	**1**	0	0	**1**
0	0	**1**	0	0	**1**	0	0	**1**	0	0
1	0	0	**1**	0	0	**1**	0	0	**1**	0
0	**1**	0	0	**1**	0	0	**1**	0	0	**1**

$\alpha = 4$ snakes, $\beta = 7$ co-snakes

$\bar{\alpha} = 1$ ouroboros, $\bar{\beta} = 1$ co-ouroboros

$\deg(p_*) = 4$, $\operatorname{codeg}(p_*) = 7$

$T(\mathcal{X}) = 3$, $\operatorname{Scale}(\mathcal{X}) = 84$

$$T(\mathcal{S}) = 3 = \frac{84}{4 \cdot 7 \cdot 1}$$

Current and future work in CA / automata networks

This project dealt with toggling independent sets (i.e., ECA rule 1) over the cycle graph Circ_n, using update order $\pi = 12 \cdots n$.

- Adapt this to other update orders, or to synchronous update.
- Increase the size of the spacial neighborhood (e.g., distance-2 graphs), or other graphs.
- Increase the size of the temporal neighborhood (e.g., CAs with memory).
- Apply these ideas to other Elementary Cellular Automata rules.
- Further study the idea of commuting complementary pairs.
  - Define the rules first, then study the dynamics.
  - When do rules arise from a Boolean network? (Complexity of determining this?)
  - When are periodic tables generated by such rules?
  - Non-commuting complentary pairs?
- Study the sum vectors in other ECA rules.
- Study concepts such as homomesy in automata networks.

Current and future work in dynamic algebraic combinatorics

This project dealt with toggling independent sets (i.e., ECA rule 1) over the cycle graph Circ_n, using update order $\pi = 12 \cdots n$.

- Apply these ideas to toggling other combinatorial objects (e.g., dominating sets).
- Develop a theory of block-sequential toggling (order ideals, antichains, noncrossing partitions, etc.).
- Look for commuting complementary pairs in other toggle actions.


HAPPY
BASTILLE DAY
•14th july•

HAPPY 80TH BIRTHDAY!

AND BELIEVE ME, NO ONE CAN WISH YOU A BETTER HAPPY BIRTHDAY THAN ME!


Happy Birthday MotherF*cker!


LET US EAT
CAKE!