On the simulation notion

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CNRS, Marseille

A controversial concept

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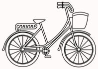
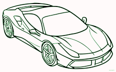
Main idea

A simulates B if A can perform the stuff that B can perform.



$\succeq$



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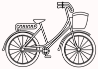
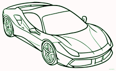
 $\succeq$ 

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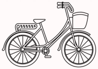

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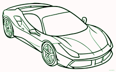

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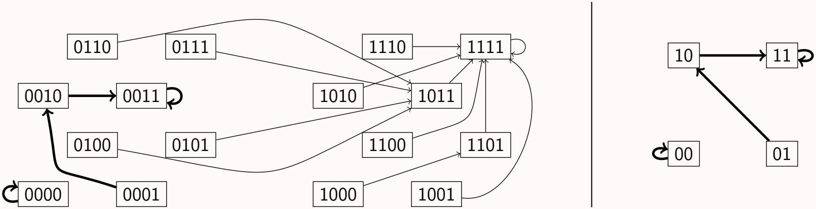
useful:

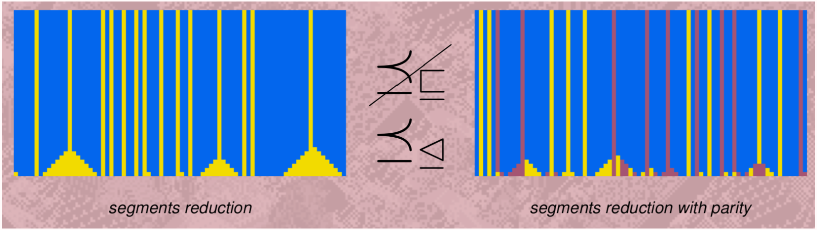
If A simulates B, then A is at least as complex as B.

Essential subsystem

dynamical system: a phase space X and a self-map $F : X \to X$.



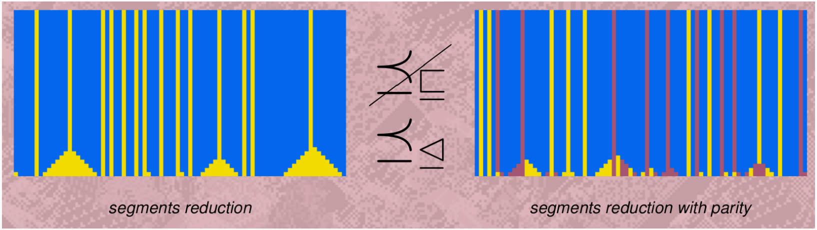
Factor



segments reduction

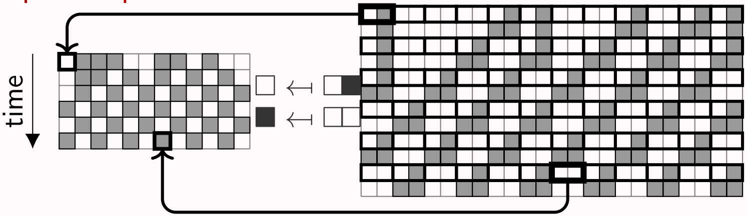
segments reduction with parity

Factor



Subsystem $\perp$ Factor [Theyssier 05,G.-Meunier-Theyssier 10]

Multiple steps



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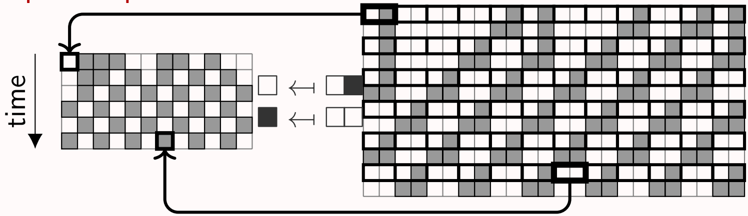

variant: also make multiple steps in the simulated side
[Rapaport 98, Ollinger 02,Theyssier 05]

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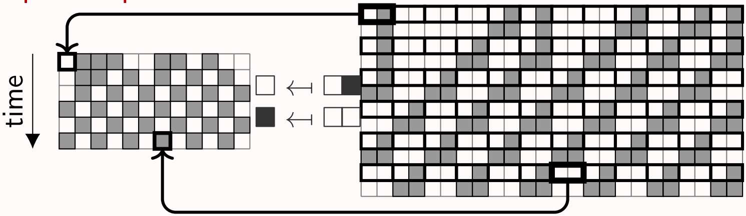

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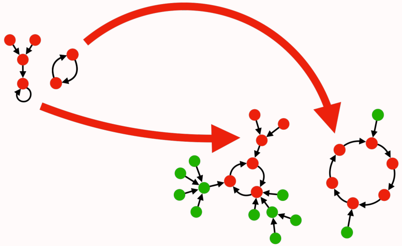
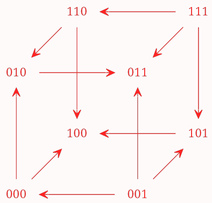


Nondeterminism



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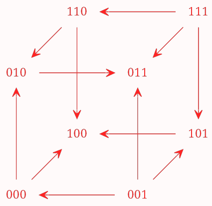

dynamical system: graph $(V, E \subset V^2)$.

e.g.: nondeterministic Turing machine

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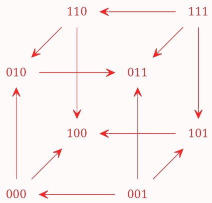

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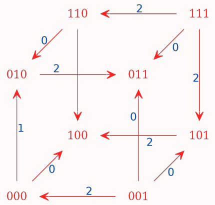
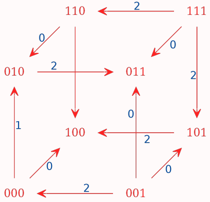


Labeled nondeterminism

dynamical system: graph
$(V, E \subset V^2 \times \Sigma)$,
with $|\Sigma| < \infty$.



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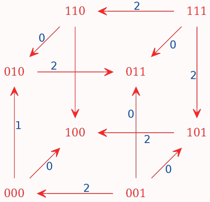

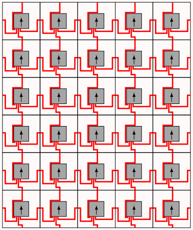

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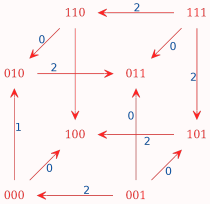

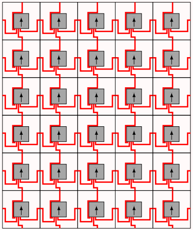

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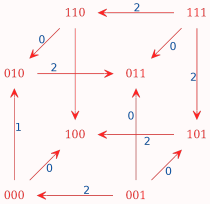

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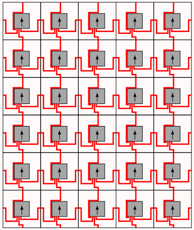
- group actions. . .



Different models

X and Y need not be the same space

e.g. simulation of Turing machines by piecewise-affine maps

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$$u_0 \text{ --- } u_1 \text{ --- } u_2 \text{ --- } u_3 \longrightarrow u_4$$

$E^{<*>}$: set of shortcuts

simulation of (V', E') by (V, E): (Z, Π, χ) where:

- Z: subgraph of $(V, E^{<*>})$;
- Π: epimorphism from Z into (V', E');
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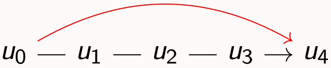

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- nice $V' \subset V$ (e.g. $V = V_1 \times V_2$ and $V' = V_1 \times \{0\}$)

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Topology

Let $\Phi : \{0,1\}^{\mathbb{Z}} \to \mathbb{R}$ a bijection, and $F : \{0,1\}^{\mathbb{Z}} \to \{0,1\}^{\mathbb{Z}}$.
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Then $(\mathbb{R}, \Phi F \Phi^{-1}) \succeq (\{0,1\}^{\mathbb{Z}}, F)$ ... unless we consider topology:

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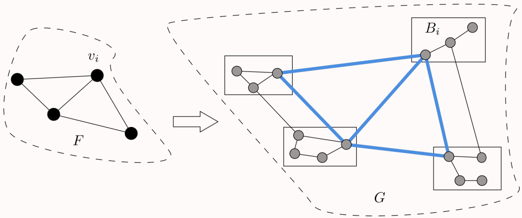
in tiling simulation: continuous $\implies$ nice ($\sim$ CA)

in Automata Networks



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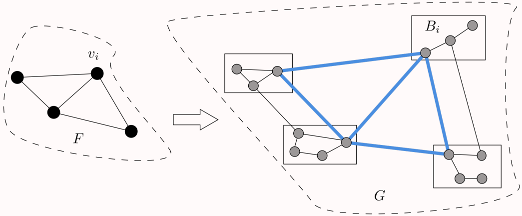

f

$1 \leftrightarrow 2$

g

$1 \leftarrow 2 \leftarrow 3$

2 3 1

x $\quad g^3(x)$ $\quad g^{3,1}(x)$ $\quad g^{1,2,3}(x)$

$$\varphi \circ g^w = f \circ \varphi$$

Monotonicity of complexity

Metatheorem: Consider a notion C of complexity.
If (Z, Π, χ) is a simulation of a Y by X, then:

$$C(Y) \leq C(X) + C(Z) + C(\Pi) + C(\chi)$$

Proof.
Compute Y by computing X and the simulation. $\square$

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e.g. P- or NP-completeness, Π^0_1- or Σ^0_3-completeness...

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Homework:
Take the simulation implicit in your last article.
Which constraints does it satisfy in the big framework?
Which constraints were needed to prove the targetted result?