Complexities of block-sequential update modes

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Joint works with Bridoux, Gaze-Maillot, Sené and Venturini

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Boolean networks and bloc-sequential updates

$$f : \{0,1\}^n \to \{0,1\}^n$$
$$\equiv$$
$$f_i : \{0,1\}^n \to \{0,1\} \text{ for } i \in [n]$$

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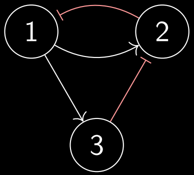

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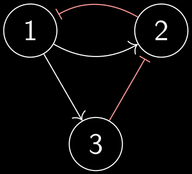
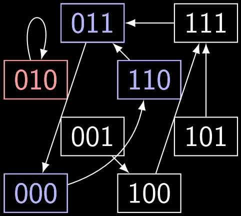


Block-sequential = ordered partition of $[n]$

eg. $(\{1,2,3\})$ or $(\{2\},\{1,3\})$ or $(\{3\},\{1\},\{2\})$...



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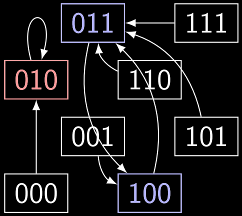

$(\{2\},\{1,3\})$

Outline

$\mathtt{BS}_n = \{$ ordered partitions of $[n]$ $\}$ (deterministic)

- ▷ Limit dynamics and block-sequential updates
- ▷ Counting bloc-sequential updates
- ▷ Extra: on computing the interaction digraph

Encoding of $f : \{0,1\}^n \to \{0,1\}^n$ as n circuits $: \{0,1\}^n \to \{0,1\}$

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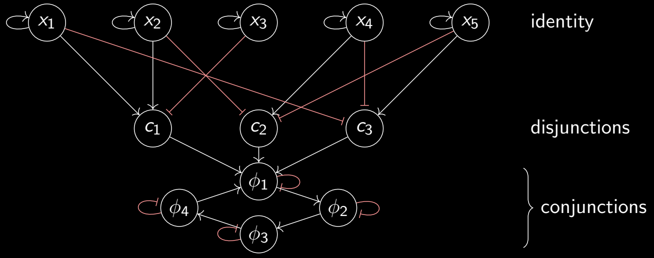
$\exists x : f^k(x) = x \wedge \forall k' < k : f^{k'}(x) \neq x$



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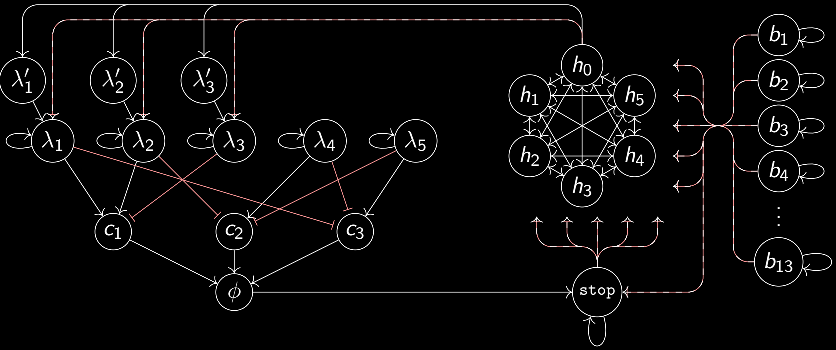

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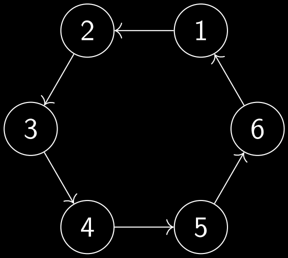

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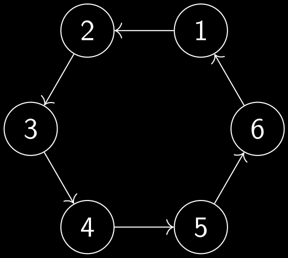

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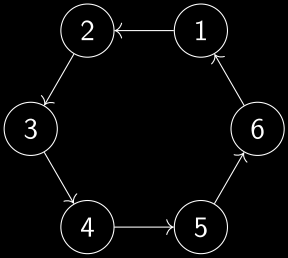

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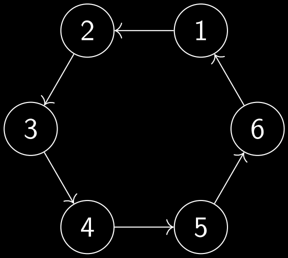

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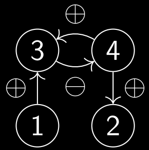
Given f and $B = (B_1, \ldots, B_p) \in \mathtt{BS}_n$, the update digraph is a $\{\oplus, \ominus\}$-edge-labeling of the interaction digraph

$$i \overset{\oplus}{\to} j \quad \iff \quad t_i \geq t_j \text{ with } i \in B_{t_i} \text{ and } j \in B_{t_j}$$
$$i \overset{\ominus}{\to} j \quad \iff \quad t_i < t_j \text{ with } i \in B_{t_i} \text{ and } j \in B_{t_j}$$

Theorem [Aracena et al. 2009]. If the update digraphs are identical then the dynamics are identical.



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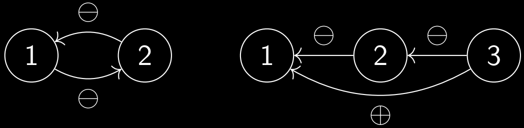
It defines an equivalence relation $B \equiv_G B'$ relative to G.

Forbidden cycles

Caution: not all edge-labelings are valid.

eg.



Theorem [Aracena et al. 2011]. An edge-labeling is valid iff the multi-digraph obtained by reversing the orientation of ⊖-arcs does not contain a cycle with at least one ⊖-arc (forbidden).

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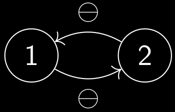

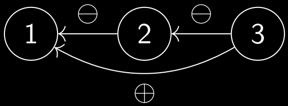

Theorem [Aracena et al. 2011]. An edge-labeling is valid iff the multi-digraph obtained by reversing the orientation of $\ominus$-arcs does not contain a cycle with at least one $\ominus$-arc (forbidden).

#UD *Input :* A digraph G on n vertices. *Ouput :* Count valid $\{\oplus, \ominus\}$-edge-labelings, ie. $\mid \mathrm{BS}_n / \equiv_G \mid$.

#P-completeness

#P= class of problems counting the number of accepting branches, of a non-deterministic Turing machine halting in polytime.
= counting the number of certificates of a problem in NP.
(solutions)

$$\{x : \exists y : R(x,y)\} \in \mathrm{NP} \iff x \mapsto |\{y : R(x,y)\}| \in \#\mathrm{P}$$

#P-completeness is relative to parcimonious polytime reductions,
preserving the number of certificates.

eg. #3-SAT, #Clique, #VertexCover, #FAS are #P-complete,
2-SAT∈ P but #2-SAT is #P-complete.

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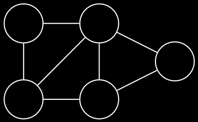

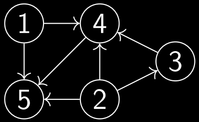

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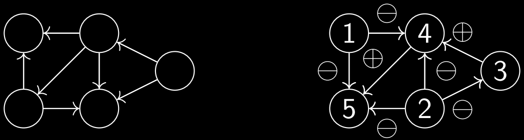

orientation $\leftrightarrow$ edge-labeling

cycle $\leftrightarrow$ forbidden cycle

More and perspectives

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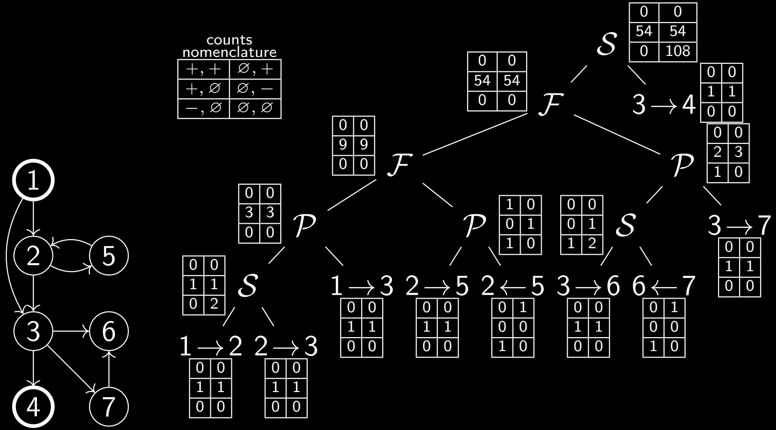

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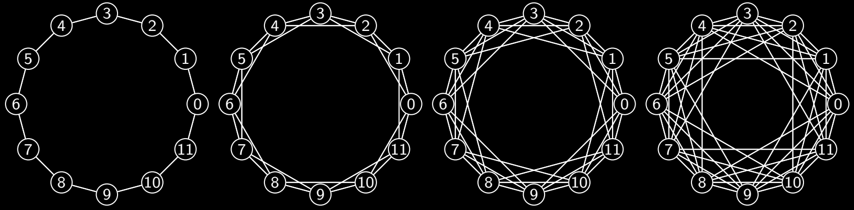
Open. Is **#UD** in FP for bounded treewidth? for Halin graphs?

Open. Count on cycles of size n with neighborhood of radius r?
For $r = 1$ we have ECAs on periodic configs: $3^n - 2^{n+1} + 2$.



Corollary. Count $T_G(2, 0)$ when G is a acyclic.

Theorem [Aracena et al. 2013]. Count $n!$ $\iff$ G is a tournament.

Extra: on computing the interaction digraph

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Interaction digraph
Input : a Boolean automata network f and a digraph G.
Ouput : is G the interaction digraph of f?

Intuition: deciding the existence of an arc is NP-complete

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$\mathrm{D^P} = \{L_1 \cap L_2 : L_1 \in \mathrm{NP} \text{ and } L_2 \in \mathrm{coNP}\}$
Polytime many-one reduction from **SAT-UNSAT**

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